1 2 1 Quadratic
1 3 3 1 Cubic
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 21 7 1

Sum of sq. of roots $P_1 - 2P_2$

P_1 products one at a time added together
P_2 " 2 " " " " "
P_3 " 3 " " " " "

Polynomial Equations
1. Divide by rational factors
2. Graph
3.

$-P_1 = \alpha + \beta + \gamma + \delta$
$+P_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$
$-P_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$
$+P_4 = \alpha\beta\gamma\delta$

Elementary Symmetric Expressions

Factor
Graph
Calculate roots

Descartes - founder of Modern Science

Symmetric expression - any two expressions changed and still remains same

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FRESHMAN ALGEBRA FOR LIBERAL ARTS C
3 4211 000087764

Hg Pk
200
4944

Virginia M

1115 a

WITHDRAWN

R. Van Dyke

Divide by key numbers
Written as many times
as their are rows less two
14 rows Key no is 12th power

$3x+5y=27$
$2x+7y=29$

$$x=\frac{\begin{vmatrix}27 & 5\\ 29 & 7\end{vmatrix}}{\begin{vmatrix}3 & 5\\ 2 & 7\end{vmatrix}}$$

$$y\begin{vmatrix}3 & 27\\ 2 & 29\end{vmatrix}$$

$$\begin{vmatrix}3 & 5\\ 2 & 7\end{vmatrix}$$

Algebra

Theory of number

determinant

Theory of Equations

Domain wanted solve factorize

Define - put in its class & then qualify it

Number - an idea you have of groups of things

difference in number in a group

Descriptive definition

Integers 1-2-3-4-etc or natural numbers

Fractions

Rational - cannot be written as a fraction

Equation -

algebraic numbers { concerned in kind of no. not size

Reals { + −

Complex {

$0 \cdot x = 0$

$\frac{5}{2}x = \frac{50}{3}$

$x = \frac{50}{3} \cdot \frac{2}{5}$

$x = \frac{20}{3}$

$3x + 5y = 27$

$2x + 7y = 29$

$5x + 12y = 56$

or $x - 2y = -2$

$3x + 5y = 27$

$2x + 7y = 29$

$3.7x + 5.7y = 27.7$

$-2.5x - 5.7y = -5.29$

$(3.7 - 2.5)x = 27.7 - 29.5$

$x = \frac{27.7 - 29.5}{3.7 - 2.5}$

$1.1x = 4.4$

$x = 4$

$x-3$

$x-5$

x^2-3x

$-5x+15$

$x^2-8x+15$

$x-6$

x^3-8x^2+15x

$-6x^2+48x-90$

$x^3-14x^2+63x-90$

$x+4$

$x^4-14x^3+63x^2-90x$

$4x^3-56x^2+252x-360$

$x^4-10x^3+7x^2+162x-360$

$x^4+5x^3+25x^2+90x+360$

$x+5$

$x^3-5x^2-18x+72$

$x^4-5x^3-18x^2+72x$ | 0

$(x-3)^3$

$1 \quad 3 \quad -6 \quad -72$

$-2 \quad -24 \quad 0$ | $x-8$

$-4 \quad +24$

-4

$1 \quad -6$

FRESHMAN ALGEBRA

For Liberal Arts Courses

BY

JAMES BYRNIE SHAW

UNIVERSITY OF ILLINOIS

$x^4-15x^2+10x+24$

$(x-2)(x-3)(x+4)(x+1)$

1 +0 −15 +10 +24 | 2

2 4 −22 −24

1 2 −11 −12 | 0 | 3

3 15 12

15 4 | 0 | −4

−4 −4

1 +1 | 0

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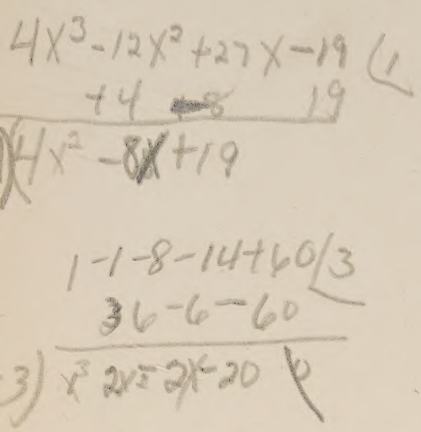

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to the
STUDENTS
who made this book possible.

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PREFACE

This text is written from a standpoint that differs considerably from that of most of the books on algebra of the present day. It undertakes definitely to base the study of Algebra upon its Beauty and not upon its Utility. The author is of the opinion that a proper classification of Mathematics would put it with the other Fine Arts. He has frequently urged that the argument for keeping it in a prescribed course be not based at all on either Utility nor a supposed unusual value for Logic, but upon its training of the Intuition and for its Æsthetic value. This is the basis of the courses for which this book is written. The author has had for some years classes consisting only of girls, and the text is substantially the course given in one semester of the Freshman year, five hours per week.

The book is not intended to contain everything that is said or done in the class. The author assumes that the class is in charge of a teacher who is interested enough to bring in any extra material, problems, comments of a philosophical character, comments on Art and Life in general, which may illuminate the fundamental thesis that here we are studying a world which is invisible but yet is essentially beautiful. That such a course is not only interesting to almost every girl but becomes even fascinating, is the experience the author has found with his classes.

The text is used very slowly, the progress being entirely at such rate as turns out to be the natural one for the class. It can, however, be easily covered in one semester. The teacher will find few problems from physics, chemistry,

engineering, accountancy, statistics, home economics, or other "applied" mathematics. If these are the end of the course, this is scarcely the book to be used. It seems to the author that such courses as those based on "practical life" are to the real study of Mathematics much as Commercial Art is to Fine Art. He is not undervaluing the practical to over-rate the other, he is merely saying there is a proper place for each.

The author is accustomed in his classes to read certain things which furnish a background for this course, and which throw light upon the æsthetic value of the subject. These readings contain a few poems carefully selected. The list varies, but will be furnished to anyone interested. Among other readings there will be mentioned a few of special interest. One is Neihardt's "Poetic Values." Almost everything he says of Art is true of Mathematics. Another is a selection of Chapters VI, VII, IX, XIII, XIV, XV from Mary Austin's "Everyman's Genius." The teacher can easily find others of a similar nature.

The use of poetry to furnish a background, rather than the constant insistence on Utility, has furnished some very interesting results, which each may discover for himself. Such poetry should be like the following from Tennyson:

> For truly, as thou sayest, a Fairy King
> And Fairy Queens have built the city, son;
> They came from out a sacred mountain-cleft,
> Toward the sunrise, each with harp in hand,
> And built it to the music of their harps.
> And as thou sayest, it is enchanted, son,
> For there is nothing in it as it seems,
> Saving the King; though some there be that hold
> The King a shadow and the city real;
> Yet take thou heed of him, for so thou pass
> Beneath this archway, then wilt thou become
> A thrall to his enchantments, for the King
> Will bind thee by such vows as is a shame

A man should not be bound by, yet the which
No man can keep; but so thou dread to swear,
Pass not beneath the gateway, but abide
Without among the cattle of the field,
For, an ye heard a music, like enow
They are building still, seeing the city is built
To music, therefore never built at all,
And therefore built forever.

Gareth and Lynette

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[Ahmes - 1550 B.C.
1150 B.C]

Euclid collected forms

CHAPTER I

THE HISTORY OF ALGEBRA

1. Mathematics in some form has a very long history.[1] As far back as 4874 B.C. a calendar was constructed showing considerable knowledge of arithmetic. This date is far back in the stone age, and long before there was an alphabet. As we come up through the succeeding ages we find woven into the historical records evidences of an ever-increasing use of arithmetic. Later in Greece we find geometry developed, and with its progress an increased understanding of the ideal character of mathematics. For mathematics is really a creation of the mind, — a creation produced perhaps under the stimulus of the human struggle for existence in the environment we find about us, but nevertheless a purely ideal construction, which during the progress of human history has become a magnificent structure. Algebra is a very important part of this structure. Geometry and the theory of functions are two other very important parts. In this sense algebra includes arithmetic as one of its divisions.

2. If we limit ourselves to what we usually understand algebra to be, we find the first treatise was written by Diophantos, of Alexandria.[2] He lived somewhere near the period of Constantine, and little remains of his books. His name is preserved, however, in the *Diophantine Equations*, — equations, for instance, which must be solved in whole numbers, or integers only. About the time the Saxons were getting settled in England, Aryabhata in

[1] The student should consult Smith: *History of Mathematics.*

[2] *Diophantos of Alexandria*, T. L. Heath.

India wrote a book on algebraic topics, and some centuries later, when Mohammed was founding a new religion, Brahmagupta of India wrote an algebra. In both these books the quadratic was solved. Two centuries later, when Haroun Al Rashid (Aaron the Just) had built Bagdad, the City of Peace, and made it the intellectual center of the world, about the time he sent to Charlemagne the first clock ever seen in Europe, Mohammed ben Musa, Al Khowarezmi, of Khorassan, wrote a treatise whose title, *Al jebr w'al muqabalah*, has furnished the name *Algebra* to our subject. The title means "transposing and reducing." This was the first systematic treatment which was not also an arithmetic. We may say that the early history of algebra came to a close about two centuries later, with Omar Khayyam (the Tentmaker),[3] a mathematician, astronomer, poet, and philosopher, far better known by his *Rubaiyat* (Quatrains), but who wrote the best algebra in existence up to his time. He lived about the time Bagdad fell to the invading Tatars, the time of William the Conqueror, a period when many changes started. He did much for astronomy, publishing new tables and reforming the calendar in a most successful way. He is a fine example of the broadest type of mathematician, interested in all the manifestations of beauty, interested in applying mathematics, interested in the serious consideration of life and the world, as well as interested in his addition to the structure of mathematics. His philosophy is particularly that of mathematics, since he finds, after examining various other solutions of the mystery of life and death, that Eternal Youth, with its adventurousness, its enthusiasm, its spontaneity, its joy of living, is the ideal way of life. And certainly, of all the products of

[3] *The Three Students*, Haldane McFall.

human thought, none other so well exhibits the character of eternal youth as does Mathematics.

3. We may mention the algebra of Bhaskara, about the time of Henry II, of Saladin, of the crusaders. After this there is a period of little advance till the time of Edward VI, of the Reformation, of De Soto's search for a new El Dorado. In 1545 appeared Cardan's *Ars Magna*. This was the beginning of a new period in the history of algebra, for it contained solutions of the cubic, and the quartic, used complex numbers, and opened new vistas. Cardan was indebted to Ferrari and Tartaglia for some of his work, however. A little after Shakespeare's time, in the time of Richelieu, lived Descartes,[4] one of the greatest of the mathematicians and philosophers. One of his most magnificent triumphs was to show the substantial unity of algebra and geometry. He added much to algebra.

4. In the closing part of the eighteenth century and the opening of the nineteenth many names become important in the history of algebra. We might mention Lagrange, Gauss, Ruffini, and Abel, who proved that in general the quintic cannot be solved by radicals only. However, the brilliant name of this period is Galois,[5] a youth of stormy career, to whom is due the modern theory of equations. He laid open to the light the essential structure of an equation and showed that the solution of an equation depends upon the solution of an accompanying "Galois resolvent equation," and also exhibited the conditions under which this latter depended upon the solution of simpler equations of the resolvent type. Almost contemporaneous with Galois was Sylvester,[6] who

[4] *René Descartes*, Mrs. E. S. Haldane.

[5] *Annales de l'École Normale*, (3) 13 (1896), pp. 197–266; P. Dupuy, *La via d'Évariste Galois.*

[6] *Johns Hopkins Alumni Magazine*, 9 (1926) 2, pp. 119–140, "J. J. Sylvester: A Brilliant and Eccentric Mathematician," David Blondheim.

extended many parts of algebra. He was another example of the mathematician-poet type, and in many of his addresses referred to mathematics as fine art. Another of the same type was Sir W. R. Hamilton,[7] who also opened a wide new field for the development of algebra by the creation of Quaternions. He was a contemporary of Galois. The nineteenth century writers on algebra were mainly concerned with exploring the new territory.

5. In this rapid and very brief survey of the history of algebra we have been able merely to refer to some of the great names in its development.[8] We may notice next the usual divisions of the subject. These may be taken conveniently as:

I. Arithmetic. Chapter II. Chapter IV. Chapter VI.
II. Equations. Chapter III. Chapter V.
III. Identities. Chapter VII.
IV. Invariants. Chapter VIII.
V. Operators. Chapter VIII.
VI. Hypernumbers. Chapter IX.

We will consider first some necessary notions from arithmetic.

[7] *Sir William Rowan Hamilton*, R. P. Graves.

[8] The student should read these biographies, at least in part, and hand in for each a paper that expresses his own opinion or reaction towards the mathematician. He will be interested also to read some of the history of algebra in various books.

math - study of ideal constructions and their properties
Alg. study of ideal structures

CHAPTER II

SOME PROPERTIES OF INTEGERS

1. The first numbers to be studied were the integers, that is, the whole numbers : 1, 2, 3, 4, · · ·. Even in Pythagoras' time, about 500 B.C., a great deal of attention was given to their study, and Pythagoras even undertook to make intergral numbers give an explanation of the universe. This attempt failed, though at the present time we find a revival of the fundamental idea, in the explanation offered by physicists for the constitution of atoms of various elements, and in the structure of energy. But important as these facts are, no one would assert that they are adequate to explain all the properties of matter and energy.

2. Numbers were early divided into the odd, and the even, the even numbers being exactly divisible by 2, the odd not divisible by 2 without remainder. The number 0 (in algebra a number, though not a number in arithmetic) is counted as even. The even numbers then are 0, 2, 4, 6, 8, · · · and the odd numbers 1, 3, 5, 7, 9, · · ·. We shall represent an integer by a capital letter, as N, so that we may express any even number by the formula $2N$, and an odd number by the formula $2N + 1$.

3. The attempts to divide numbers soon led to the distinction of numbers as prime or composite. A prime number can be divided only by 1 and the number itself. A composite number has other divisors. For instance, the numbers

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,
43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

are all the prime numbers under 100.

There is an unproved theorem, called Goldbach's theorem, which says that every integer is situated half-way

between two primes, in at least one way, and often half-way between several pairs of primes. For instance, 28 is half-way between 19 and 37, 13 and 43, 3 and 53. This implies that the list of primes is endless. It was known to Euclid that there is no end to the list of primes. No formula is known which will always give a prime number, though there are some quite notable ones that give several primes. For instance, one due to Euler is $x^2 + x + 41$, which will give a prime for all values of x from 0 to 39 inclusive. Another is $N^2 + N + 17$, which will give primes for $N = 0, 1, \cdots 15$. Also $2N^2 + 29$, which gives a prime for $N = 0, \cdots 28$.

The student should notice that in these instances we have a case of the difference between the induction of science and mathematical proofs. Although Goldbach's theorem holds true for some ten thousand numbers tried, the mathematician cannot consider that it is proved for all cases. Likewise for the formulae just given for primes. Mere observation of individual cases, no matter how numerous, does not constitute mathematical proof, though often it will suggest theorems which are true.

4. The only simple method of factoring numbers is to start with the small primes in order, 2, 3, 5, 7, etc., and by trial find a divisor, if there is any, then continue in the same manner with the quotient. For instance, we find

3	999999
3	333333
7	111111
11	15873
13	1443
3	111
	37

Therefore $999999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$. The order of the divisions is immaterial.

It is evident that simple tests as to divisibility are useful. The following are the ones commonly in use.

a. A number is divisible by 2 when its last figure[1] can be divided by 2.

b. A number is divisible by 5 when its last figure is divisible by 5.

c. A number is divisible by 3 when the sum of its figures is divisible by 3.

d. A number is divisible by 9 when the sum of its figures is divisible by 9.

e. A number is divisible by 11 when the sum of the alternate figures and the sum of those omitted have a difference divisible by 11.

For instance, 798 is divisible by 2 and by 3, 715 is divisible by 5 and by 11.

There are some other tests also useful:

f. A number is divisible by 4 when its last two figures make a number divisible by 4; and it is divisible by 8 if the last three figures, by 16 if the last four figures, etc., are divisible by the same numbers.

g. A number is divisible by 25 if the last two figures are divisible by 25; by 125 if the last three are divisible by 125; by 625 if the last four, etc.

h. A number is divisible by 7 if when it is cut into two numbers by cutting off the last figure, the difference of the last figure multiplied by 2 and the other part of the number can be divided by 7. For instance, 64,666 is divisible by 7, the reductions being made as shown

$$\begin{array}{r} 6466/6 \\ \underline{\quad 12} \\ 645/4 \\ \underline{\quad 8} \\ 63/7 \\ \underline{14} \\ 49 \end{array}$$ which can be divided by 7.

[1] We shall use "figure" to indicate either the character or the number it represents.

i. If we cut off the last figure, multiply by 4, and add to the other part of a number, the number is divisible by 13 if the result obtained can be divided by 13. For instance, 11,999 is divisible by 13, the mental reduction being

11999
36
1235
20
143 which is divisible by 13.

j. If $N = 10\,a + b$, then N is divisible by 17 when $a - 5\,b$ is divisible by 17. N is divisible by 19 when $a + 2\,b$ is divisible by 19. N is divisible by 23 when $a + 7\,b$ is divisible by 23. For 29 the form is $a + 3\,b$. For 31 the form is $a - 3\,b$. For instance, 392,863 is divisible by 19, 23, 29, and 31.

392863	392863	392863	392863
39292	39307	39295	39277
3933	3979	3944	3906
39	460	406	372
57		58	31

EXERCISES

Prove by inspection that the following numbers are divisible by those written after them:

1. 144,739 by 7, 23, 29, 31.
2. 69,223 by 7, 11, 29, 31.
3. 1,480,479 by 3, 7, 11, 17, 29, 13.

5. All these and many more similar theorems follow from the properties of the number 10 and its powers. For the common method of writing numbers on the decimal system is a method of detaching coefficients from powers of 10 and writing only the coefficients. For instance, 96,287 really means

$$9 \cdot 10^4 + 6 \cdot 10^3 + 2 \cdot 10^2 + 8 \cdot 10 + 7.$$

In the usual manner of writing, all the powers of 10 have been omitted. However, for many purposes we must remember they are still to be considered as if they were written. For example, to decide what is the "order" of the number above we have to notice that the left-hand figure 9 belongs to 10^4, hence the order is 4. The order of the left-hand figure of a number is called its *characteristic.* The orders are conveniently shown in the scheme

$$9\,8\,7\,6\,5\,4\,3\,2\,1\,0\,.\bar{1}\,\bar{2}\,\bar{3}\,\bar{4}\,\bar{5}\,\bar{6}\,\bar{7}\,\bar{8}\,\bar{9},$$

the orders to the right being negative. For instance, 78,932,645 has characteristic 7, while .000087631 has characteristic -5.

If now we notice that $10 = 9 + 1$, $10^2 = 99 + 1$, $10^3 = 999 + 1$, etc., we can see at once that the number $A\,10^s + B\,10^{s-1} + C\,10^{s-2} + \cdots + N$ when divided by 9 will give remainders $A \quad + B \quad + C \quad + \quad + N$. If this sum is divisible by 9, the original number is divisible by 9. Similar proofs furnish the other theorems above. The ambitious student can easily discover them.

6. The student will find it convenient to recognize the various powers of the integers. The following table contains the first five powers.

	1	2	3	4	5
1	1	1	1	1	1
2	2	4	8	16	32
3	3	9	27	81	243
4	4	16	64	256	1024
5	5	25	125	625	3125
6	6	36	216	1296	7776
7	7	49	343	2401	16807
8	8	64	512	4096	32768
9	9	81	729	6561	59049

It will be observed that the terminal figure of a square is either 1, 4, 5, 6, or 9. The terminal figure for a cube is the same as for the number cubed if it ends in 1, 4, 5, 6. If numbers ending in 2 or 8 are cubed, the terminal figure will be 8 or 2 respectively, and for those ending in 3 or 7 it will be 7 or 3. Fourth powers end only in 1, 5, or 6. Fifth powers end with the same figure as the number itself. For powers higher than five we can find the terminal figure by dividing the power by 5 and keeping the remainder. The terminal figure will then be given by the column above for the remainder. For instance, 19th powers will end in the same figures as 4th powers. Powers of numbers ending with 5 or 6 will end with 5 or 6. Odd powers of numbers ending in 4 or 9 will end in 4 or 9, and the even powers will end with 6 or 1.

7. A table of numbers, called Pascal's [2] triangle, is useful later. It is found thus:

					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1
	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1

Each number in a line below the first is the sum of the two numbers just above it on either side. For instance, $56 = 35 + 21$. These numbers are called *binomial coefficients* because they are the coefficients in the powers of a binomial such as $(a + b)$. For instance,

$$(a + b)^6 = a^6 + 6\,a^5b + 15\,a^4b^2 + 20\,a^3b^3 + 15\,a^2b^4 + 6\,ab^5 + b^6.$$

[2] His studies were published in 1665, after his death. The student should build up the lines by multiplying $a + b$ by itself.

In Art. 9 below a formula will be given for them from which any one may be found directly, without finding the others.

It is interesting also to notice that any number is the sum of the entire diagonal either to its right or its left, as far down as the last number just above the line in which the number selected lies. For instance,

$$56 = 21 + 15 + 10 + 6 + 3 + 1.$$
$$70 = 35 + 20 + 10 + 4 + 1.$$
$$28 = 7 + 6 + 5 + 4 + 3 + 2 + 1.$$

These binomial coefficients have many other interesting properties. Some of these will be noticed later.

8. We consider next the method of changing the representation of a number on one scale to that on another. For instance, suppose we desire to represent 254,361 on the scale of ten by a number on the scale of 7. We first divide the given number by 7, finding the quotient 36,337 and remainder 2. Therefore

$$254{,}361 = 36{,}337 \cdot 7 + 2.$$

Dividing 36,337 by 7 we find $36{,}337 = 5191 \cdot 7 + 0$. Dividing 5191 in the same way we find

$$5191 = 741 \cdot 7 + 4.$$

Then $741 = 105 \cdot 7 + 6$, $105 = 15 \cdot 7 + 0$, $15 = 2 \cdot 7 + 1$. From these results, substituting back from each into the preceding we have

$$254{,}361 = 2 \cdot 7^6 + 1 \cdot 7^5 + 0 \cdot 7^4 + 6 \cdot 7^3 + 4 \cdot 7^2 + 0 \cdot 7 + 2.$$

That is, 254,361 on the scale of 10 is 2,106,402 on the scale of 7. On the scale of 12 it would be 103,249. On the scale of 2 it is 111,110,000,110,011,001.

9. We will introduce a convenient notation which has frequent use. We abbreviate products of the integers, starting with 1, by using an exclamation mark !, read "factorial." We have in this notation

1 factorial: $1! = 1$.
2 factorial: $2! = 1 \cdot 2 = 2$.
3 factorial: $3! = 1 \cdot 2 \cdot 3 = 6$.
4 factorial: $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.
5 factorial: $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.
6 factorial: $6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$, etc.

The numbers in Pascal's triangle can all be written in this way, as we can see most easily by considering the product $(x + a)^n$. Let n first be 4, then we multiply $(x + a)^4 = (x + a)(x + a)(x + a)(x + a)$. We will write down the terms directly. The first will come from taking x in each factor and will be x^4. The next will come by taking x three times and one a. As there are four ways we can do this, it gives $4\,x^3a$. The next will come by taking two x's and two a's, which can be done in $\frac{4 \cdot 3}{1 \cdot 2}$ ways.[3] So proceeding the full form would be

$$x^4 + \frac{4}{1}x^3a + \frac{4 \cdot 3}{1 \cdot 2}x^2a^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}xa^3 + a^4.$$

By inserting the necessary factors in numerator and denominator we write this

$$\frac{4!}{4!}x^4 + \frac{4!}{3!\,1!}x^3a + \frac{4!}{2!\,2!}x^2a^2 + \frac{4!}{1!\,3!}xa^3 + \frac{4!}{4!}a^4.$$

In the general case we may condense the writing of the binomial formula thus

$$(x + a)^n = \Sigma \frac{n!}{r!(n - r)!}x^r a^{n-r},$$

where the Σ means the sum of all terms answering the description that follows, and where r takes the values $0, 1, 2, 3, \cdots, n$.

[3] The student should write down the twelve forms made from four letters, two at a time, as from p, q, r, s: pq, pr, ps, qp, qr, etc. Then change p and q to x; r and s to a. He will see then every form occurs just twice. So for the other terms.

Notice the numbers in the denominator are the exponents on x and a. The numbers $\frac{n!}{r!(n-r)!}$ are the numbers of the nth line in Pascal's triangle. If 0! occurs, it is taken to mean 1.

EXERCISES

1. Reduce 2,106,402 on the scale of 7 back to the scale of 10. The divisor is 13.

2. If we consider the figures 123456 as a number on the scale of 7, reduce it to a number on the scale of 10. Also if we consider the same figures as a number on the scale of 8, of 9, and of 11, in each case reduce to a number on the scale of 10.

3. Expand by using Pascal's triangle

$$(a+b)^4$$
$$(x+y)^5$$
$$(c+d)^6$$

4. Find $\quad 7!\ \ 8!\ \ 9!\ \ 10!\ \ 11!\ \ 12!$

$$\frac{12!}{5!7!} \qquad \frac{12!}{3!4!5!}$$

CHAPTER III

ELIMINATION

1. Early in the history of the world man came across problems that showed him he could not get along with integers only. For instance, in such a problem as the division of two loaves of bread among three persons, the solution of what we would write as $3x = 2$, it is necessary to have the idea of fraction. The early world found considerable difficulty in handling the fractions they came across, but still there grew up some knowledge of means to find the answers to questions like the one above. In brief, the set of fractions were created to supplement the whole numbers or integers. Since we call such numbers *ratios*, the entire list is called the *domain of rational numbers*. This domain includes integers, for there is no difference between 4 and $\frac{8}{2}$, or $\frac{20}{5}$, etc. Even in Pythagoras' time the idea of rational number was in evidence.

We may consider then that the simplest type of equation is

$$ax = b.$$

Its solution is

$$x = \frac{b}{a}. \qquad (a \neq 0).$$

If a and b are themselves rational numbers, since any fraction divided by a fraction gives a fraction, x can always be found as a fraction.

Equations that involve x only to the power 1, as above (for x and x^1 mean the same thing), are called equations of *first degree*, or more commonly, linear equations.

If there are several terms involving x, as $4x - 3x + 7x + 12x = 28 - 5x + 2x$, these must be collected into a single term before solving. This equation reduces to $20x = 28 - 3x$. This is the "muquabala" of Al Khowarezmi. We now add $3x$ to both sides, which gives us $23x = 28$. This is "aljebr." Finally, we have $x = \frac{28}{23}$. In the reduction it should be noted that in $4x - 3x + 7x + 12x$ we have x as a common factor of every term, so that the first reduction is to write the set as

$$x(4 - 3 + 7 + 12).$$

Then the numbers in the parenthesis may be combined to give 20. This method of reduction is very common in algebraic problems. It is found first in arithmetic in the addition of fractions. For instance, to add

$$\tfrac{3}{4} + \tfrac{5}{6} + \tfrac{6}{7}$$

we change each fraction into a fraction whose denominator is divisible by 4, 6, 7. The smallest number that will do is 84. The only change permitted in a fraction is to have its numerator and denominator multiplied by the same number. To change 4 to 84 needs multiplication by 21, hence $\frac{3}{4} = \frac{63}{84}$. The multipliers for the other numbers are easily seen to be 14 and 12, giving finally as an equivalent set

$$\tfrac{63}{84} + \tfrac{70}{84} + \tfrac{72}{84}.$$

Now the common factor, $\frac{1}{84}$, may be written down first and the other numbers put together in a parenthesis, thus

$$\tfrac{1}{84}(63 + 70 + 72) = \tfrac{205}{84}.$$

In handling algebraic fractions precisely similar work is done.

EXERCISES

Solve for x in each case.

1. $4 - 2x = 8 - 6x$.
2. $3x + 6 - 2x = 7x$.
3. $ax + bx = a^2 - b^2$.
4. $\frac{x}{2} + \frac{x}{5} = 5$.

5. $(x-2)(x-7)+(x+1)(x-3)=8x+2(x-8)(x-7)-2.$

6. $acx - bdx = ck - bm.$

7. $(x+a)(x+b)=(x+c)(x+d).$

8. $(p-q)x=(p+q)x-r.$

9. $\frac{1}{x-3}+\frac{1}{x-6}=\frac{1}{x-4}+\frac{1}{x-5}.$

2. The equations next in simplicity are those which involve two unknown numbers, x and y, each only to the first degree, and not multiplied together. For example, consider

$$3x + 5y = 27,$$
$$2x + 7y = 29.$$

In order to solve these equations we must arrive at an equation containing x only, and another containing y only. To do this we must change these equations into two others in which the x terms have the same coefficient or the y terms have the same coefficient. This can be done by multiplying the first equation[1] by 7, the second by -5. This will give us the pair

$$21x + 35y = 189,$$
$$-10x - 35y = -145.$$

It is clear now that if we add the two expressions on the left, the y terms will cancel out, and only x will be left, so that we have

$$11x = 44, \qquad x = 4.$$

Similarly, if we multiply the first by -2, the second by 3, we have

$$-6x - 10y = -54,$$
$$6x + 21y = 87.$$

By adding we find the x terms have been eliminated, and we have

$$11y = 33, \qquad y = 3.$$

[1] "Multiplying an equation" is short for "multiplying each term of the equation."

3. In order to study what takes place in an algebraic process it is better not to have definite numbers in the problem, for these combine together to hide the essence of the process. Consequently we will take as our equations

$$ax + by = e,$$
$$cx + dy = f.$$

If we multiply the first by d and the second by $-b$ and add, we shall eliminate y.

$$adx + bdy = ed,$$
$$-bcx - bdy = -bf.$$

And adding these together we have by collecting the two x terms in a parenthesis:

$$(ad - bc)x = ed - bf,$$
$$x = \frac{ed - bf}{ad - bc} \quad \text{if } (ad - bc \neq 0).$$

Again, multiplying the first equation by $-c$ and the second by a and adding, we have

$$(ad - bc)y = af - ec,$$
$$y = \frac{af - ec}{ad - bc} \quad \text{if } (ad - bc \neq 0).$$

These two expressions have some very notable features. The first we see is that the denominators are the same. Secondly, the numerator of x differs from the denominator only in that we have e in place of a and f in place of c, while the numerator of y has e in place of b and f in place of d. Further the three different expressions are all alike in that each consists of the difference of two terms, each term with two factors, and if we examine where the letters stood in the original equations, we find that no two numbers are multiplied together which stood in the same row or the same column. This feature is important, and we can make it more evident as well as the other facts

we stated, if we invent a new notation for these forms. This we do by writing for $ad - bc$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$; for $ed - bf$, $\begin{vmatrix} e & b \\ f & d \end{vmatrix}$; for $af - ec$, $\begin{vmatrix} a & e \\ c & f \end{vmatrix}$. And when we see such a form, we understand that it means we are to multiply the two numbers on the diagonal from the upper left corner to the lower right corner, and from the product subtract the product of the other two numbers. It is to be noted that the coefficients of x and y make up the denominators, but to make up the numerators we substitute in the denominator forms the column of numbers on the right side of the equations for the column of coefficients belonging to the unknown we are finding. If we go back to our numerical problem, it would read now

$$x = \frac{\begin{vmatrix} 27 & 5 \\ 29 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & 7 \end{vmatrix}} = \frac{189 - 145}{21 - 10} = 4,$$

$$y = \frac{\begin{vmatrix} 3 & 27 \\ 2 & 29 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & 7 \end{vmatrix}} = \frac{87 - 54}{21 - 10} = 3.$$

Forms that can be written in this new notation are called *determinants*. They occur frequently in mathematical investigations and are of high importance. In writing the values of x and of y the denominators should be constructed first by copying the block of coefficients on the left (which, of course, have been systematically arranged).

EXERCISES

Solve by writing down the fractions directly for x and for y, and reducing.

1. $2x + 3y = 32$ $\qquad$ $3x + 2y = 28.$
2. $3x + 2y = 9$ $\qquad$ $5x - 2y = 13.$

3. $4x - 3y = 11$ $\qquad 2x + 7y = 13.$

4. $5x - 8y = -46$ $\qquad 3x - 5y = -29.$

5. $8x - 3y = 11$ $\qquad 5x + 7y = -24.$

6. $lx + my = p$ $\qquad rx + sy = q.$

7. $ax + by = ad + bc$ $\qquad cx + dy = 2cd.$

8. $(a+b)x+(a-b)x=5a-b$ $\qquad (a-b)x-(a+b)x=-a-5b.$

9. Reduce the determinants by writing them out and reducing:

$$\begin{vmatrix} 14 & 17 \\ -5 & -9 \end{vmatrix} \quad \begin{vmatrix} a+b & c+d \\ a-b & -c+d \end{vmatrix} \quad \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} \quad \begin{vmatrix} 8 & 24 \\ 5 & 15 \end{vmatrix}$$

4. It is evident in the case of a single equation and single unknown, x, as

$$ax = b,$$

that we may find trouble, since we have to make a division, and division by zero is not permitted. If $a = 0$, then we have $0x = b$. There is evidently no solution at all unless $b = 0$. In this case x may have any value, since 0 times any number gives 0.

So too in the case of two unknown numbers, we have trouble if the denominator of the fractions above is zero. That would mean $ad - bc = 0$, or $a : c = b : d$. Just as for one unknown, unless the numerators are also zero, the equations have no solution at all. But we may have

$$0x = ed - bf = 0, \qquad 0y = af - ec = 0.$$

In these equations x and y are left indefinite, and we may choose any value we like for either of them. The other is connected by each of the two original equations, and if we assume a value for one, we must find the value for the other out of one of the equations. For instance, if we have

$$3x + 8y = 27,$$
$$6x + 16y = 54,$$

we find all three determinants come out 0. But if we choose $x = 1$, the first equation shows us that $y = 3$, if we choose $x = 2$, $y = 2\frac{5}{8}$, etc.

5. If we consider the determinant

$$\begin{vmatrix} b & a \\ d & c \end{vmatrix}$$

we see by writing it out in full, $bc - ad$, that it is the negative of

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Hence if we interchange the two columns, we change the sign of the value. The same is true if we interchange the two rows, as the student may easily prove.

Again the determinant

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

has the same value as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

so that interchanging the rows and columns does not affect the value. These properties of determinants, we shall see later, are always true for any determinant.

6. We take up now the problem of three equations and three unknown numbers, for instance,

$$\begin{aligned} 3x + 2y + 4z &= 28, \\ 5x + 3y + 2z &= 27, \\ 4x + 5y + 3z &= 35. \end{aligned}$$

We surmise here, based upon our experience with the preceding case, that we have to find multipliers for the three equations such that when added all the unknown numbers are eliminated except a single one. That such

multipliers exist for this problem is easy to see, for if we multiply the first by -1, the second by 14, the third by -8, and add, the coefficients of y and z will come out 0. We have

$$\begin{array}{rrrrr} -3x & -2y & -4z & = -28 & \\ 70x & +42y & +28z & = 378 & \\ -32x & -40y & -24z & = -280 & \\ \hline 35x & +0y & +0z & = 70 & x = 2. \end{array}$$

Again, if we multiply the first by -7, the second by -7, the third by 14, we have

$$\begin{array}{rrrrr} -21x & -14y & -28z & = -196 & \\ -35x & -21y & -14z & = -189 & \\ 56x & +70y & +42z & = 490 & \\ \hline 0x & +35y & +0z & = 105 & y = 3. \end{array}$$

Finally, if we multiply the first by $+13$, the second by -7, the third by -1, we have

$$\begin{array}{rrrrr} 39x & +26y & +52z & = 364 & \\ -35x & -21y & -14z & = -189 & \\ -4x & -5y & -3z & = -35 & \\ \hline 0x & +0y & +35z & = 140 & z = 4. \end{array}$$

The multipliers are found by a method similar to that in the case of two unknown numbers. However, they are themselves determinants. For instance, in the elimination of y and z, the multiplier of the first equation is

$$\begin{vmatrix} 3 & 2 \\ 5 & 3 \end{vmatrix}$$

This is built up from the coefficients of y and z just as they stand in the rows except the first. The multiplier of the second equation is

$$-\begin{vmatrix} 2 & 4 \\ 5 & 3 \end{vmatrix}$$

which is made up of the coefficients of y and z just as they

stand in the rows except the second. So the third multiplier is

$$\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$$

It will be noticed that the second multiplier has its sign changed.

In order to make a study of the problem for any coefficients we proceed in a similar manner. But to save writing, and for convenient arrangement we define now what a block of numbers like

$$\begin{vmatrix} 3 & 2 & 4 \\ 5 & 3 & 2 \\ 4 & 5 & 3 \end{vmatrix}$$

will be understood to mean.

Its definition we will make thus, using the numbers in the first column as coefficients:

$$\begin{vmatrix} 3 & 2 & 4 \\ 5 & 3 & 2 \\ 4 & 5 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & 2 \\ 5 & 3 \end{vmatrix} - 5\begin{vmatrix} 2 & 4 \\ 5 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$$

This would give for its value $3(-1) - 5(-14) + 4(-8) = 35$. This is the coefficient of x after adding the multiplied equations. It is evident we may write the coefficient of y as

$$\begin{vmatrix} 2 & 2 & 4 \\ 3 & 3 & 2 \\ 5 & 5 & 3 \end{vmatrix}$$ and of z as $$\begin{vmatrix} 4 & 2 & 4 \\ 2 & 3 & 2 \\ 3 & 5 & 3 \end{vmatrix}$$, each reducing to zero.

We notice that these two that reduce to zero have two columns just alike. We shall see that this is sufficient proof that the value is zero. Further, the determinant we have from the right-hand numbers is

$$\begin{vmatrix} 28 & 2 & 4 \\ 27 & 3 & 2 \\ 35 & 5 & 3 \end{vmatrix}$$

which reduces to 70 as we saw above. The value of

x could have been written down directly thus in this notation:

$$x = \frac{\begin{vmatrix} 28 & 2 & 4 \\ 27 & 3 & 2 \\ 35 & 5 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 4 \\ 5 & 3 & 2 \\ 4 & 5 & 3 \end{vmatrix}}$$

This form is evidently the desired extension of the form in the problem for two unknown numbers. We shall find the proper similar forms for y and z. Before we do this we must notice some properties of the determinants.

If we interchange the y and z columns in the original set of equations, we know they are not altered and will give the same values for the unknown letters. But since interchanging two columns in the determinants of two rows and columns, that is, determinants of order 2, changed the sign of the value, it is clear that if we find x from the new arrangement, the multiplier of x, which becomes the denominator above, will be

$$\begin{vmatrix} 3 & 4 & 2 \\ 5 & 2 & 3 \\ 4 & 3 & 5 \end{vmatrix} \text{ which must equal } - \begin{vmatrix} 3 & 2 & 4 \\ 5 & 3 & 2 \\ 4 & 5 & 3 \end{vmatrix}$$

Hence we see that interchanging the second and third columns changes the sign in a determinant of third order just as in one of second order. Since a similar change would take place in the numerator determinant, x would come out with the same value. Again, if we were to interchange in the set of equations the first two columns, we will not change the values of x and y, but the determinants will be different now since the coefficient for y will come out

$$\begin{vmatrix} 2 & 3 & 4 \\ 3 & 5 & 2 \\ 5 & 4 & 3 \end{vmatrix}$$

This differs from the denominator for x only in having the first two columns interchanged. The coefficient for y, however, which we found by the multiplications, was

$$-2\begin{vmatrix}5&2\\4&3\end{vmatrix}+3\begin{vmatrix}3&4\\4&3\end{vmatrix}-5\begin{vmatrix}3&4\\5&2\end{vmatrix}=-\begin{vmatrix}2&3&4\\3&5&2\\5&4&3\end{vmatrix}$$

And this turned out to be the same as the coefficient for x, and so we have at least

$$\begin{vmatrix}3&2&4\\5&3&2\\4&5&3\end{vmatrix}=-\begin{vmatrix}2&3&4\\3&5&2\\5&4&3\end{vmatrix}$$

That is, in this determinant interchanging the first two columns has changed the sign. This leads us to think we have a general property, that any two columns in a determinant may be interchanged if we change the sign of the determinant. If this is true, as we shall prove, then we may keep the original order of the columns for all the denominators, and for the numerators all we have to do is to erase the column of the denominator which corresponds to the unknown number we wish to find, and insert in its place the column of numbers from the right side. We may write then for this problem, finally, if we set

$$\Delta=\begin{vmatrix}3&2&4\\5&3&2\\4&5&3\end{vmatrix}$$

$$\Delta\cdot x=\begin{vmatrix}28&2&4\\27&3&2\\35&5&3\end{vmatrix}\qquad\Delta\cdot y=\begin{vmatrix}3&28&4\\5&27&2\\4&35&3\end{vmatrix}\qquad\Delta\cdot z=\begin{vmatrix}3&2&28\\5&3&27\\4&5&35\end{vmatrix}$$

7. We will consider the problem now for any coefficients. In the first place we will give a different definition of the meaning of an array of numbers in a square form, with the two vertical lines on the sides. This definition

will cover determinants of any order. We will use a notation which will indicate where each number came from. For instance, c_4 is in the third column and fourth row.

DEFINITION. The determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & \cdots & k_1 \\ a_2 & b_2 & c_2 & d_2 & \cdots & k_2 \\ a_3 & b_3 & c_3 & d_3 & \cdots & k_3 \\ a_4 & b_4 & c_4 & d_4 & \cdots & k_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & d_n & \cdots & k_n \end{vmatrix}$$

means the value found by reducing an expression which is called the *expanded form of the determinant*. The expanded form is made up as follows:

Every possible term is written down which consists of the product of one a, one b, one c, one d, $\cdots$, out to one k, and as subscripts on these letters one 1, one 2, one 3, one 4, $\cdots$, out to one n. These terms are then given signs according to the number of letters, or number of subscripts, or number of both, which are inverted in order, that is, stand in front of another that should in the natural arrangement precede. For every inversion, the sign is made to contain a minus. An even number of inversions brings out the final sign, $+$, an odd number, $-$. The terms of this expression are then combined by adding or subtracting, as far as possible. If they are all definite numbers, the final result is a single number, which is the value of the determinant.

It is evident that either the letters may be kept in order, or the subscripts may be kept in order. The other set will then take every arrangement possible for a set of n objects. It is necessary to see that if the signs of the terms are found for any given arrangement they will not change by rearranging the factors of the term even though

this introduces other inversions, or removes inversions, for the number changed will always be even.

8. We will first write out the expanded form for

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Keeping the letters in order, we have the terms $a_1b_2c_3$, $a_1b_3c_2$, $a_2b_1c_3$, $a_2b_3c_1$, $a_3b_1c_2$, $a_3b_2c_1$, and no other forms are possible. In the first term everything is in its natural order. In the second we see 3 in front of 2, one inversion, so the sign of the second term will be $-$. In the third, 2 in front of 1, sign $-$. In the fourth both 2 and 3 in front of 1, two negatives, sign $+$. In the fifth 3 in front of 1 and 2, sign $+$, and in the last 3 in front of 1 and 2, and 2 in front of 1, sign $-$.

This gives

$$a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

as the expanded form for a determinant of third order. We notice there are $3 \cdot 2 \cdot 1$ terms, half plus, half minus. If now we write the factors of any term in a different order, interchanging them two at one time, we see that whether the letters were in natural order or inverted, the new arrangement is the opposite. Likewise whether the subscripts were in natural order or inverted the new arrangement is opposite, hence we have introduced two reversals of the orders, and this being an even number will leave no change of sign. Consequently the order of the factors is immaterial for determinants of order 3. It is easy to see in the general form

$$a_\alpha b_\beta c_\gamma d_\delta \cdots k_\kappa,$$

where the Greek letters are the numbers 1, 2, 3, 4, $\cdots n$ in some order, that if we interchange any two factors we

introduce two oppositions to what we had,[2] which, being even, makes no change. Hence we were warranted in the definition to permit the order to be arbitrary, knowing that when the total number of inversions is counted the sign will come out all right. For instance above, if we wrote the next to the last term

$$c_2 a_3 b_1$$

we have in the letters two inversions, in the numbers two, a total of four, which is even, and the sign should be $+$, as it is.

9. Since in the expanded form no term has more than one a, there is but one number from the first column in any term. If we collect all those that have a_1, we see that the coefficient of a_1 is a form which has one b, c, d, $\cdots$ k in every term, and one 2, 3, 4, $\cdots$ n in every term, and that the law of signs is the same as we gave for a determinant, hence the coefficient of a_1 will be the determinant

$$\begin{vmatrix} b_1 c_1 d_1 & \cdots & k_1 \\ b_2 c_2 d_2 & \cdots & k_2 \\ \cdots & \cdots & \cdots \\ b_n c_n d_n & \cdots & k_n \end{vmatrix}$$

In the same way we find that the coefficient of a_2 in the expanded form will be the determinant produced from the original form by erasing the column for a and the row for 2, also with a minus sign introduced into every term, since every term will lose a variation by starting

[2] There is a set of subscripts between the two, consisting of three groups: (1) the subscripts less than either of the two interchanged, (2) the subscripts greater than either of the two interchanged, (3) the subscripts between the two interchanged. Moving the two concerned will not affect the inversions due to (1) or (2), and will increase or decrease the inversions by twice the number in (3). Hence the effective change is due to inverting the two moved.

with this new determinant. Determinants made from another determinant by erasing columns and the same number of rows are called minors of the original determinant. If we indicate the minor left when we erase the a column and the i row by A_i, then we can write the expanded form in this manner

$$= a_1A_1 - a_2A_2 + a_3A_3 - \cdots (-)^{n-1}a_nA_n.$$

This includes for determinants of order 3 the first definition given of them. It is more general, however, for we could write out the expansion collected as to any other column, as for instance b, as

$$= - b_1B_1 + b_2B_2 - b_3B_3 + \cdots.$$

This property shows us that the method of expressing the various determinants involved in the equations of three unknown numbers and the fractional forms for these unknown numbers was quite correct, and that it is a general theorem for sets of n equations involving n unknown numbers.

10. We see also that if in every term we leave the numbers standing, but interchange two given letters, then we have introduced an odd number of changes and if we do this throughout the expanded form, we have changed every sign. But interchanging two letters means in the square array that we have interchanged two columns. Therefore interchanging any two columns will change the sign of the determinant. The same is true if we interchange two rows. We now have the general proof of a property as noticed above in the special cases.

An important consequence is that if a determinant has two rows alike (or two columns) its value is zero. For, interchanging the rows will change its sign, but the inter-

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change does not change the values in the rows and columns: hence we have

$$\Delta = -\Delta, \text{ and } \Delta = 0.$$

11. If we go back now to the expanded form arranged according to the elements of a column as

$$a_1A_1 - a_2A_2 + a_3A_3 - \cdots a_nA_n,$$

we see that if all the numbers $a_1, a_2, \cdots$ have a common factor this may be written in front and we have a new determinant whose column corresponding to the a-column has been divided by the factor. Suppose the factor of all the a's is m, so that

$$a_1 = mv_1, \qquad a_2 = mv_2, \cdots,$$
$$\text{then } m(v_1A_1 - v_2A_2 + \cdots) = m\Delta',$$

where Δ' differs from Δ only in having the numbers v in place of the numbers a. As for example

$$\begin{vmatrix} 4 & 7 & 6 & 5 \\ 8 & 1 & 2 & 9 \\ 6 & 5 & 3 & 1 \\ 12 & 9 & 6 & 3 \end{vmatrix} = 2 \begin{vmatrix} 2 & 7 & 6 & 5 \\ 4 & 1 & 2 & 9 \\ 3 & 5 & 3 & 1 \\ 6 & 9 & 6 & 3 \end{vmatrix}$$

As a similar theorem is true for the rows, we can factor 3 out of the last row, giving

$$2 \cdot 3 \begin{vmatrix} 2 & 7 & 6 & 5 \\ 4 & 1 & 2 & 9 \\ 3 & 5 & 3 & 1 \\ 2 & 3 & 2 & 1 \end{vmatrix}$$

Also if the numbers appear in the forms

$$a_1 = p_1 + q_1, \qquad a_2 = p_2 + q_2, \qquad a_3 = p_3 + q_3, \cdots,$$

then we can separate the terms and have

$$= (p_1A_1 - p_2A_2 + p_3A_3 -) + (q_1A_1 - q_2A_2 + q_3A_3 - \cdots)$$
$$= \Delta' + \Delta'',$$

where Δ' differs from Δ only in having for the column of a's the column of p's, and Δ'' has for tne column of a's the column of q's. For example,

$$\begin{vmatrix} 7 & 5 & 3 & 1 \\ 6 & 2 & 4 & 3 \\ 9 & 1 & 3 & 2 \\ 8 & 4 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 7-7 & 5 & 3 & 1 \\ 6-21 & 2 & 4 & 3 \\ 9-14 & 1 & 3 & 2 \\ 8-35 & 4 & 1 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 5 & 3 & 1 \\ 21 & 2 & 4 & 3 \\ 14 & 1 & 3 & 2 \\ 35 & 4 & 1 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 5 & 3 & 1 \\ -15 & 2 & 4 & 3 \\ -5 & 1 & 3 & 2 \\ -27 & 4 & 1 & 5 \end{vmatrix} + 7 \begin{vmatrix} 1 & 5 & 3 & 1 \\ 3 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 5 & 4 & 1 & 5 \end{vmatrix}$$

Now the last determinant is zero since two columns are alike. Hence we find the first determinant is equal to another whose columns are the same as in the first, except one, in this case the first, and we have a 0 at the top of this new column. This shows us that we may add or subtract one column from another, getting a new determinant whose value, however, will be the same as that of the original determinant. This is true of any row as well as of any column. It enables us to reduce a determinant of a given order to one of the next lower order. For instance, by continuing with the determinant above, we may change it into

$$\begin{vmatrix} 7 & 5 & 3 & 1 \\ 6 & 2 & 4 & 3 \\ 9 & 1 & 3 & 2 \\ 8 & 4 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 7-7, & 5-5, & 3-3, & 1 \\ 6-21, & 2-15, & 4-9, & 3 \\ 9-14, & 1-10, & 3-6, & 2 \\ 8-35, & 4-25, & 1-15, & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ -15 & -13 & -5 & 3 \\ -5 & -9 & -3 & 2 \\ -27 & -21 & -14 & 5 \end{vmatrix}$$

But in the final form the minors the 0's multiply will give 0 whatever they are, so that the only minor which

remains, times -1, will give the same value; that is, we cancel out the last column and top row, bringing down the 1 with its sign changed, and writing the determinant of order 3, thus

$$-1\begin{vmatrix} -15 & -13 & -5 \\ -5 & -9 & -3 \\ -27 & -21 & -14 \end{vmatrix}$$

The 1 has its sign changed owing to its position, for if we count up the inversions we find the number is odd. We now have the original determinant reduced from 4 rows and columns to an equal value, but given by a determinant with three rows and columns. We may change all signs in this, since that makes three changes on account of the column and one for the $-$ outside, giving

$$\begin{vmatrix} 15 & 13 & 5 \\ 5 & 9 & 3 \\ 27 & 21 & 14 \end{vmatrix}$$

To reduce this in a similar manner is evidently not so easy as it stands, for the multipliers would have to be fractional. However, we can subtract 2 times the third column from the first, giving

$$\begin{vmatrix} 5 & 13 & 5 \\ -1 & 9 & 3 \\ -1 & 21 & 14 \end{vmatrix}$$

And now if we add the last row multiplied by 5 to the top row, and subtract the last row from the second row, we will have

$$\begin{vmatrix} 0 & 118 & 75 \\ 0 & -12 & -11 \\ -1 & 21 & 14 \end{vmatrix} = -1\begin{vmatrix} 118 & 75 \\ -12 & -11 \end{vmatrix} = 398.$$

In order to systematize this procedure, and to take care of cases in which there is no row or column with an element

which will divide exactly all the others in either its row or its column, we will consider the following determinant, the procedure in this case being perfectly general, as may be seen.

$$\Delta = \begin{vmatrix} 5 & 4 & 3 \\ 2 & 7 & 9 \\ 9 & 4 & 8 \end{vmatrix}$$

We first make it possible to divide every element of the first row by 5, the left-hand element, by multiplying the elements in the second and third columns by 5, and then dividing the new determinant by 5^2 which will restore its value.

$$\Delta = \frac{1}{5^2} \begin{vmatrix} 5 & 20 & 15 \\ 2 & 35 & 45 \\ 9 & 20 & 40 \end{vmatrix}$$

We may now multiply the first column by 4 and subtract it from the second, then multiply the first by 3 and subtract it from the third, giving

$$= \frac{1}{5^2} \begin{vmatrix} 5 & 0 & 0 \\ 2 & 27 & 39 \\ 9 & -16 & 13 \end{vmatrix} = \frac{5}{5^2} \begin{vmatrix} 27 & 39 \\ -16 & 13 \end{vmatrix} = \frac{945}{5} = 189.$$

We notice that the number in the upper left corner, called the key number, appears as divisor as many times as the number of columns less one; but as it also comes out in the reduction as a multiplier once, it is actually a divisor to a power which is the number of columns less two. Further, it must divide exactly, since the original expansion contains only multiplications and additions and thus could give only integers from integers. This gives us a check on the accuracy of the work. As another example we will reduce a determinant of the fourth order, writing only the necessary results. To simplify the work we will make a border cutting off the top row and the left column, and then we will notice that each element in the

reduced form will stand in the place corresponding to the element in the small square in the original form which it came. For instance,

$$\left|\begin{array}{cc|cc} 3 & 2 & 5 & 7 \\ 2 & -3 & 4 & 6 \\ -5 & 2 & 6 & 8 \\ 4 & 5 & -3 & 5 \end{array}\right| = \frac{1}{3^2}\begin{vmatrix} -13 & 2 & 4 \\ 16 & 43 & 59 \\ 7 & -29 & -13 \end{vmatrix}$$

To get -13 we have the block $\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix}$, to get 2 we have $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}$, to get 4 we have $\begin{vmatrix} 3 & 7 \\ 2 & 6 \end{vmatrix}$. In the next line to get 16 we have $\begin{vmatrix} 3 & 2 \\ -5 & 2 \end{vmatrix}$, to get 43 we have $\begin{vmatrix} 3 & 5 \\ -5 & 6 \end{vmatrix}$, and to get 59 we have $\begin{vmatrix} 3 & 7 \\ -5 & 8 \end{vmatrix}$. In the last line to get 7 we have $\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}$, to get -29 we have $\begin{vmatrix} 3 & 5 \\ 4 & -3 \end{vmatrix}$, and to get -13 we have $\begin{vmatrix} 3 & 7 \\ 4 & 5 \end{vmatrix}$. In all these the key, 3, is the multiplier of the numbers in the square fenced off, while the border numbers multiply only each other. We can tell where any one result came from by inspection, as for example, suppose we wish to check 59. It must have come from 8 in the square, and we have $\begin{vmatrix} 3 & 7 \\ 5 & 8 \end{vmatrix} = 59$. Continuing we have

$$= \frac{1}{9}\left|\begin{array}{c|cc} -13 & 2 & 4 \\ 16 & 43 & 59 \\ 7 & -29 & -13 \end{array}\right| = \frac{1}{9}\cdot\frac{1}{-13}\begin{vmatrix} -591 & -831 \\ 363 & 141 \end{vmatrix}$$

$$= \frac{218322}{9 \cdot 13} = 1866.$$

The same procedure would enable us to reduce determinants of any order.

12. It does not make any difference of course if the elements in the determinant are letters. For instance,

$$\begin{vmatrix} x & a & b \\ -a & x & c \\ -b & -c & x \end{vmatrix} = \frac{1}{x}\begin{vmatrix} x^2+a^2 & cx+ab \\ -cx+ab & x^2+b^2 \end{vmatrix}$$

$$= \frac{x^4 + a^2x^2 + b^2x^2 + c^2x^2}{x}$$

$$= x^3 + (a^2 + b^2 + c^2)x.$$

The key number, of course, cannot be 0, since it has to be used as a divisor. In case the upper left number is 0, or if it is larger than others in the first row, it simplifies matters to interchange some other column with the first.

Of course any element may be taken to be the key number, but the most convenient one to take is the upper left-hand element, and next to that the lower right-hand element. For example,

$$\begin{vmatrix} 4 & 3 & 2 \\ 5 & 4 & 3 \\ 3 & 2 & 8 \end{vmatrix} = \frac{1}{4}\begin{vmatrix} 1 & 2 \\ -1 & 26 \end{vmatrix} = 7,$$

$$\begin{vmatrix} 4 & 3 & 2 \\ 5 & 4 & 3 \\ 3 & 2 & 8 \end{vmatrix} = \frac{1}{8}\begin{vmatrix} 26 & 20 \\ 31 & 26 \end{vmatrix} = 7.$$

13. We may now return to the problem of solving three simultaneous equations for x, y, z. For example,

$$4x + 3y + 2z = 4,$$
$$5x + 4y + 3z = 5,$$
$$3x + 2y + 8z = 17.$$

As we have just seen $\Delta = 7$. We have then

$$7x = \begin{vmatrix} 4 & 3 & 2 \\ 5 & 4 & 3 \\ 17 & 2 & 8 \end{vmatrix} = 21, \qquad x = 3.$$

$$7y = \begin{vmatrix} 4 & 4 & 2 \\ 5 & 5 & 3 \\ 3 & 17 & 8 \end{vmatrix} = -28, \qquad y = -4.$$

$$7z = \begin{vmatrix} 4 & 3 & 4 \\ 5 & 4 & 5 \\ 3 & 2 & 17 \end{vmatrix} = 14, \qquad z = 2.$$

EXERCISES

Solve for the unknown numbers.

1. $2x + y + 3z = 1, x + 2y + 4z = -3, 3x + y + 2z = 6.$
2. $5x - 2y + 3z = 5,\ 3x + 4y - z = 21,\ 2x - y - 3z = -3.$
3. $4x - 3y - z = 0, 5x + 2y - 6z = 1, 7x - y - 5z = 1.$
4. $2x + 3y + 5z + 4w = -7,\ 3x - 5y - 2z + 3w = -1,$
   $6x + 7y + 3z - w = 11, 5x - 4y - 5z + 4w = 12.$
5. $2x + 3y + 4z = 6,\ 5x + y + 4z = 9,\ 7x + 4y + 8z = 15.$
6. $ax + by + cz = a^3 + b^3 + c^3 - 3abc, \quad bx + cy + az = 0,$
   $cx + ay + bz = 0.$

14. We could make the reductions in a little different manner, which will lead us to the next problem we shall consider. Let all terms be put on the left-hand side of the equation, as, for example,

$$\begin{aligned} 2x - 3y + 4z - 14 &= 0, \\ 3x - 5y + 7z - 24 &= 0, \\ 4x + 2y - 3z - 5 &= 0, \end{aligned}$$

If we multiply the first equation by $\begin{vmatrix} 3 & -5 \\ 4 & 2 \end{vmatrix}$, the second by $-\begin{vmatrix} 2 & -3 \\ 4 & 2 \end{vmatrix}$, and the third by $\begin{vmatrix} 2 & -3 \\ 3 & -5 \end{vmatrix}$ and add, we shall have

$$\begin{vmatrix} 2 & -3 & 4z - 14 \\ 3 & -5 & 7z - 24 \\ 4 & 2 & -3z - 5 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 0 \\ 3 & -5 & 0 \\ 4 & 2 & 0 \end{vmatrix} = 0.$$

The left side reduces to

$$\tfrac{1}{2}\begin{vmatrix} -1 & 2z - 6 \\ 16 & -22z + 46 \end{vmatrix} = -10z + 50 = 0.$$

From this $z = 5$. The value of Δ is -10. The method really does in two steps what the other method does in one step. However, there are cases where this form must be used. We take up one such problem at once.

15. To solve the equations, *not linear*,

$$x^2 + 3xy - 22 = 0, \qquad x + 2y - 8 = 0.$$

We must decide first whether we shall eliminate x or y. Since y has the lowest powers, we choose it, and arrange the equations in columns as to y. Thus

$$\begin{aligned} 3xy + x^2 - 22 &= 0, \\ 2y + x - 8 &= 0. \end{aligned}$$

If now we use the proper multipliers, we have

$$\begin{vmatrix} 3x & x^2 - 22 \\ 2 & x - 8 \end{vmatrix} = 0 = x^2 - 24x + 44.$$

This is a quadratic and we will defer its solution. In case we choose to eliminate x, we arrange thus

$$\begin{aligned} 1x^2 + 3yx - 22 &= 0, \\ 1x + 2y - 8 &= 0. \end{aligned}$$

Since there are now three columns and only two rows, we evidently cannot eliminate as before, but we will remove the difficulty if we make another equation, which we do by multiplying the second equation (the lower) by x, and inserting this equation. We have now three:

$$\begin{aligned} 1x^2 + \quad 3yx \quad - 22 &= 0, \\ 1x^2 + (2y - 8)x + \quad 0 &= 0, \\ 0x^2 + \quad 1x + 2y - 8 &= 0. \end{aligned}$$

Using proper multipliers we have then the determinant

$$\begin{vmatrix} 1 & 3y & -22 \\ 1 & 2y - 8 & 0 \\ 0 & 1 & 2y - 8 \end{vmatrix} = 0 = -2y^2 - 8y + 42.$$

If we can solve this, we can find y. In actual practice we do not have to write the extra equations. The determinant just written shows the same expressions on the third line as on the second line, except they have been set over one column. We may then make use of a rule due to Sylvester, to the effect that the coefficients of the powers of the unknown, being eliminated, will occur for each equation as many times as the degree of the other equation, being set over one column each time after starting at the first left column. The vacant places are filled with 0. When we have written coefficients down in this way, we will find that the number of columns equals the number of rows. For instance,

$$\begin{aligned} 3x^2 + 2xy + 5y^2 - 27 &= 0, \\ 2x^2 - xy + 3y^2 - 6 &= 0. \end{aligned}$$

The Sylvester determinant, if we eliminate x, is

$$\begin{vmatrix} 3 & 2y & 5y^2 - 27 & 0 \\ 0 & 3 & 2y & 5y^2 - 27 \\ 2 & -y & 3y^2 - 6 & 0 \\ 0 & 2 & -y & 3y^2 - 6 \end{vmatrix} = 0.$$

If we eliminate y, we have

$$\begin{vmatrix} 5 & 2x & 3x^2 - 27 & 0 \\ 0 & 5 & 2x & 3x^2 - 27 \\ 3 & -x & 2x^2 - 6 & 0 \\ 0 & 3 & -x & 2x^2 - 6 \end{vmatrix} = 0.$$

As these forms usually contain a large number of 0's, their reduction is generally easy. After the resulting equation is found it must be solved by methods to be considered in the next chapter.

In selecting the letter to be eliminated it should be remembered that the number of rows and columns in the eliminant will be the sum of the two degrees of the

letter eliminated. For instance, in case we have the equations

$$x^3y + 3x^2 + 2y - 6 = 0,$$
$$x^2 + 2xy - 5 = 0,$$

the order of the determinant will be 5 if we eliminate x, while it will be only 2 if we eliminate y.

EXERCISES

Find the determinant (eliminant) which is the simplest for the following equations and reduce to a polynomial.

1. $x + y = 28$ $xy = 187.$
2. $x - y = 8$ $xy = 20.$
3. $x^2 + y^2 = 89$ $x + y = 13.$
4. $x^2 + xy - y^2 - 5 = 0$ $2x^2 - 3xy + 2y^2 - 14 = 0.$
5. $x^2 + 3xy - 28 = 0$ $4y^2 + xy - 8 = 0.$
6. $x^3 + y^3 = 35$ $x + y = 5.$
7. $x^3 - y^3 = 61$ $x - y = 1.$

16. Determinants occur so frequently in mathematical work that we will consider some with interesting characters.

The determinant

$$\begin{vmatrix} a - x & b \\ c & d - x \end{vmatrix} = x^2 - (a + d)x + \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a - x & b & c \\ d & e - x & f \\ g & h & k - x \end{vmatrix} = -x^3 + x^2(a + e + k)$$

$$- x\left(\begin{vmatrix} a & b \\ d & e \end{vmatrix} + \begin{vmatrix} a & c \\ g & k \end{vmatrix} + \begin{vmatrix} e & f \\ h & k \end{vmatrix}\right) + \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

$\begin{vmatrix} a & x \\ -x & a \end{vmatrix} = a^2 + x^2.$ If $a = 0$, this reduces to x^2.

$\begin{vmatrix} a & b & c \\ -b & a & d \\ -c & -d & a \end{vmatrix} = a^3 + a(b^2 + c^2 + d^2).$ If $a = 0$, this reduces to 0.

$$\begin{vmatrix} a & b & c & d \\ -b & a & e & f \\ -c & -e & a & g \\ -d & -f & -g & a \end{vmatrix}$$

$$= a^4 + a^2(b^2 + c^2 + d^2 + e^2 + f^2 + g^2) + (bg - cf + de)^2.$$

If $a = 0$, this expression is a square.

Determinants like the last three are called *skew-symmetric*. If the main diagonal has only 0 for its numbers, the value of the determinant is 0 if of odd order, and is a square if of even order. That it is zero for the odd case is easily seen from the fact that it is not changed in value if the rows are turned into columns and columns into rows, but this will change the sign, hence it is zero.

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = [a^2 + b^2 + c^2 + d^2]^2.$$

$$\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = (x_1 - x)(y_2 - y_1) - (x_2 - x_1)(y_1 - y).$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = -(a - b)(a - c)(b - c).$$

17. The student should notice that a determinant is a structure with a very definite design. It consists of two parts, one positive the other negative, just as many terms in one as in the other. The terms are all balanced with just one number from each of the sets in rows, and just one from each of the sets in columns. The symmetry involved in it is of a rather intricate kind, but it is of such a nature that interchanging any two sets from rows or

any two sets from columns make the positive terms negative, and the negative terms positive, which is a sort of minor symmetry. Design means order and composition. The student should notice how these are exhibited in each algebraic structure considered.

18. Arrays of numbers which do not have the same number of rows and columns are also to be found in algebra. They furnish systems of determinants which have meaning in various problems. For instance the array

$$\begin{vmatrix} a & b & c \\ l & m & n \end{vmatrix}$$

will furnish three determinants by omitting in turn each column. These would be

$$\begin{vmatrix} b & c \\ m & n \end{vmatrix} \qquad -\begin{vmatrix} a & c \\ l & n \end{vmatrix} \qquad \begin{vmatrix} a & b \\ l & m \end{vmatrix}$$

If a, b, c are the coefficients of x, y, z as in the expression $ax + by + cz$; and if l, m, n are coefficients of x, y, z, as in $lx + my + nz$, then the expression of which these three determinants are the coefficients will be of much importance in studying the two forms. Other numerous similar forms arise in the advanced study of these linear forms. The study of *linearity* is one of the most fruitful in advanced algebra.

$x^3 = 2$ Omar overdue

Oct. 25th Descartes (math 6)

Haldane

I am at least aware of my own conscious existence

CHAPTER IV

THE RATIONAL DOMAIN

1. We have seen in the preceding chapter that an equation which is linear requires for its solution only the domain of rational numbers. These are numbers which are either integers or else the ratios of integers. The sum, the difference, the product, and the quotient of any two rational numbers is a rational number. In the next chapter we shall have occasion to study irrational numbers.

2. The origin of number in the history of the human race is very obscure. The study of primitive tribes has at various times led to the conclusions that man first used counting, that he first placed things in order, as first, second, third, etc., and that he used the method of tally. In all these, however, it seems after a careful study that number itself is not a product of the material things he handled, but is of a purely mental or ideal character. Numbers are objects, in the sense that they may be studied by more than one person, all persons arriving at the same statements about numbers and their properties. They are ideal objects in the sense that they are not observed by the senses.

3. The numbers first created by man for enabling him to handle his environment were the integers or natural numbers, 1, 2, 3, · · ·. The notation for these which we use did not come into existence for many centuries. We still have a survival of other notations in the Roman method of writing, such as I, II, III, IV, · · ·. This

notation was very cumbersome for calculations, though it is possible to carry on arithmetical operations in such terms. The adoption of the scale of ten seems to have arisen from the habit of tallying on the fingers, toes, and other parts of the human body. However, a scale of sixty was in use in Chaldea, and many numerical records are written in this way. This arose perhaps from the study of astronomy. It survives in the writing of degrees, minutes, and seconds.

4. One of the important notions for the student to get is that the objects of mathematics are of a purely ideal character, and while human thought makes use of them in gaining control of the material world, they are not in a strict sense an outcome from the material world. This becomes evident at once when we consider negative numbers, since these were not only not deduced from material objects, but their introduction into mathematics took a long time. They were called fictive, or non-existent, numbers, imaginary in a strict sense. As a matter of fact, they were used in mathematics for some time before they began to find applications in human affairs. And when we remember that it is impossible to bring into a room negative three pencils, or any other material objects, we can see readily their purely ideal character. The same was true to even a more pronounced extent in the use of the number i, the "square root of minus one." We will come back to this case in the next chapter. In the mathematical study of space of four dimensions we have another conspicuous instance of the transcendence of the material world. These mathematical objects are imaginary only in the sense that they are not objects for the senses. In fact, in many places in mathematics any attempt to connect the objects studied with sense-representations will lead to very serious errors.

5. We have then in mathematics a definite and long-established case of human knowledge which has not been derived from the sense-data of human experience. The proposition sometimes met with, that ultimately all human knowledge rests upon sense-data, is seen to be certainly false. If it merely meant that the statements of human thought in language demand material means, it would be a mere triviality, but it is generally intended to have a far greater meaning. Mathematics has then something very definite to say to philosophy, psychology, and other branches of knowledge with regard to the ultimate facts of life. We will recur to this in the course of the text.

6. The question as to where mathematics should be classed in the general scheme of knowledge has already been hinted at. If we understand by art the various parts of human life which are the products of the creative imagination, such as painting, music, etc., then we can say that mathematics is one of the arts, and more exactly one of the fine arts. For it has a distinctly esthetic character. It deals with intangible, invisible, inaudible objects, and is interested primarily in the structure of these objects, particularly in the symmetry, rhythm, unity, harmony, and design which are to be found there. All these are esthetic characters.

7. In the determinants we have just studied the student can perceive a design; indeed the method of writing the constituents in a square was invented in order to make the essential design stand out more clearly. The various ways in which the square array can be rearranged without changing the determinant value, such as moving the lines, turning the array over on its diagonal, reversing every line, — bring out the underlying symmetry, which is of a rather high order.

8. There is a close connection between mathematics

and poetry.[1] We will introduce occasional references to this connection. That the connection is a little subtle does not make it the less real. The student is recommended to study this connection in the poem of Mary Brent Whiteside:

YOU SHALL SEE THE WIND[2]

One asked: "But have you seen the wind — the west wind pass?"
I heard it heralding the rain:
I heard the stirred leaves whisper through the grain,
And the long grass.
I saw the thin, outcurving ripples shake
The mirrored lilies, dreaming on the lake,
Where clouds of jewelled dragon-flies awake.
I saw a warm peach in the orchard fall;
This, when the west wind passed me; that was all.

I asked: "Is there at last some other thing to see,
Beyond the blown stalk of a rose,
Down that far pathway where the west wind goes,
In mystery?"
The answer came: "In secret robes of blue,
Woven of dusk and dawn, it passes you,
Who may not follow as we others do.
Yet one day, glimmering across the grass,
You too shall *see* the wind — the west wind pass."

[1] This has been said by many of the greatest mathematicians. Quotations may be found in Moritz: *Memorabilia Mathematica.*

[2] Reprinted by permission of *Lyric West*, Los Angeles, Calif.

CHAPTER V

EQUATIONS WITH ONE UNKNOWN NUMBER

1. In order to understand how to solve equations in one unknown number, say x, it is necessary to understand how to factor such expressions as occur. These expressions can always be reduced to *polynomials*, that is, to expressions which are of the form

$$ax^n + bx^{n-1} + cx^{n-2} + \cdots + k = 0,$$

where $a, b, c \cdots k$ are integers, and n is a positive integer. To see how these are to be factored, the best way is to construct some of them and notice the essential elements of the construction. This is one of the characteristic procedures in mathematics, a synthesis which permits the essential nature of the structure to become evident. We will then first multiply as an example

$$\begin{array}{r} x - 3 \\ x - 5 \\ \hline x^2 - 3x \\ \quad - 5x + 15 \\ \hline x^2 - 8x + 15 \end{array}$$

In this we notice that the end number, 15, is the product of the two end numbers in the factors, that is, $15 = (-3)(-5)$. This must be the case, since 15 is the only term without x in it to some power, and can come only from the terms that do not have x in them. Again, multiply this result by $x - 7$.

$$\begin{array}{l} x^2 - 8x + 15 \\ x - 7 \\ \hline x^3 - 8x^2 + 15x \\ \quad - 7x^2 + 56x - 105 \\ \hline x^3 - 15x^2 + 71x - 105 \end{array}$$

The -105 is the product of the -3, -5, -7. Multiply again by $x + 4$.

$$\begin{array}{l} x^3 - 15x^2 + 71x - 105 \\ x + 4 \\ \hline x^4 - 15x^3 + 71x^2 - 105x \\ \quad + 4x^3 - 60x^2 + 284x - 420 \\ \hline x^4 - 11x^3 + 11x^2 + 179x - 420 \end{array}$$

The end number, 420, is the product of the end numbers of the factors. This suggests that if we had given an expression of this kind to be factored we would have to try as divisors $x +$ or $x -$ such numbers as would divide the last number in the polynomial. We will carry through a division of this kind, arranging the work so that we will do no unnecessary copying.

$$\begin{array}{l} x^4 - 11x^3 + 11x^2 + 179x - 420 \\ \quad - 5x^3 + 30x^2 + 95x - 420 \\ \hline x^4 - 6x^3 - 19x^2 + 84x \end{array} \begin{array}{l} (x - 5 \\ \hline x^3 - 6x^2 - 19x + 84 \end{array}$$

We get the first term of the quotient by dividing x into x^4, which gives x^3. This is multiplied into the two terms of the divisor, and the result, $x^4 - 5x^3$, is written under the dividend. This is then subtracted and the $-6x^3$ is written down below. The remainder of the original expression, however, should not be copied. As it is already in plain sight there is no need of a copy. We now divide x into $-6x^3$, giving $-6x^2$, which is put into the quotient and multiplied into the divisor. The first term need not be written, as it merely cancels the term already down, $-6x^3$. The second term is written under

the 11 x^2 and subtracted, giving $-19x^2$. So proceeding, the other terms are found, and at the end as we subtract -420 from -420 we have 0 for remainder. We will now look over this process. We notice first that the coefficients are the essential numbers in the process, and that we do not need to write the powers of x. If we leave them out, thus *detaching the coefficients*, we have

$$\begin{array}{l} 1 - 11 + 11 + 179 - 420 \;(1 - 5 \\ \quad - 5 + 30 + 95 - 420 \;\overline{1 - 6 - 19 + 84} \\ \overline{1 - 6 - 19 + 84} \end{array}$$

The student will notice the $1x^4$ has been dropped down to be on the same line as the other canceled terms. He will notice also that now we have a line of repeated numbers, $1 - 6 - 19 + 84$, and both lines need not be written. This will always be the case when the divisor starts with $1x$, for 1 divided into any number gives the same number. Further, in the subtraction all the signs were to be changed and then the numbers added algebraically. This could be accomplished automatically if the sign of the -5 is changed. We will do this, remembering that now we are ready to add, and the work would appear thus

$$\begin{array}{l} 1 - 11 + 11 + 179 - 420 \;(5 \\ \qquad 5 - 30 - 95 + 420 \\ \overline{1 - 6 - 19 + 84,\qquad 0} \end{array}$$

That is, the quotient is $1x^3 - 6x^2 - 19x + 84$ and remainder is 0.

As a further example we will divide the above polynomial by $x - 2$.

$$\begin{array}{l} 1 - 11 + 11 + 179 - 420 \;(2 \\ \qquad 2 - 18 - 14 + 330 \\ \overline{1 - 9 - 7 + 165, \;-90} \end{array}$$

quotient $1\,x^3 - 9\,x^2 - 7\,x + 165$, and remainder -90. This form of division is called "synthetic."

2. Let us for a moment look at another question; namely, how to substitute 2 for x in this polynomial. The polynomial being $x^4 - 11\,x^3 + 11\,x^2 + 179\,x - 420$, we might make the substitution by writing 2 for x, and reducing, thus

$$2^4 - 11 \cdot 2^3 + 11 \cdot 2^2 + 179 \cdot 2 - 420 = -90.$$

But an easier method is to start at the top and substitute 2 for only one x at a time, the work running thus (remembering that $x^4 = x \cdot x^3$, $x^3 = x \cdot x^2$, $x^2 = x \cdot x$).

$$\begin{array}{r} x^4 - 11\,x^3 + 11\,x^2 + 179\,x - 420 \;(2 \\ \underline{\quad 2\,x^3 - 18\,x^2 - \;\; 14\,x + 330} \\ 1\,x \cdot x^3, - 9\,x \cdot x^2, - 7\,x \cdot x, + 165\,x, - 90 \end{array}$$

But if we leave out all the x powers, this would read

$$\begin{array}{r} 1 - 11 + 11 + 179 - 420 \;(2 \\ \underline{\quad 2 - 18 - \;\; 14 + 330} \\ 1 - \;\; 9 - \;\; 7 + 165, \; - 90 \end{array}$$

This work is now the exact duplicate of the synthetic division above. And we can see that in all such cases we would be writing the same figures, and performing the same operations. This leads to the theorem: *If a polynomial in x is divided by $x - a$, the remainder will be the same as the value of the polynomial when a is substituted in it for x.* This is called the "remainder theorem." When $x - a$ is a factor, the remainder is 0, and it follows that if a is substituted for x we have 0. This corollary is often called the "factor theorem." We will now make use of these facts to factor a new polynomial. Consider

$$x^4 + 6\,x^3 - 21\,x^2 - 74\,x + 168.$$

The divisors of 168 are 1, 2, 3, 4, 6, 7, 8, 12, etc. So we may try $x \pm 1$, $x \pm 2$, $x \pm 3$, etc. Starting with 1 as the simplest, we have

$$\begin{array}{r} 1 + 6 - 21 - 74 + 168 \,(1 \\ \underline{\quad 1 + 7 - 14 - 88} \\ 1 + 7 - 14 - 88 + 80 \end{array}$$

giving a remainder 80. Trying next 2, we have

$$\begin{array}{r} 1 + 6 - 21 - 74 + 168 \,(2 \\ \underline{\quad 2 + 16 - 10 - 168} \\ 1 + 8 - 5 - 84 \end{array}$$

This goes exactly, so that one factor is $x - 2$. We continue with the quotient, trying next 3, thus

$$\begin{array}{r} 1 + 8 - 5 - 84 \,(3 \\ \underline{\quad 3 + 33 + 84} \\ 1 + 11 + 28 \end{array}$$

This goes exactly, giving as factor $x - 3$. In the quotient we see only plus signs, so no plus number could possibly bring out 0 at the end, therefore we try next -7,

$$\begin{array}{r} 1 + 11 + 28 \,(-7 \\ \underline{\quad - 7 - 28} \\ 1 + 4 \end{array}$$

This completes the factorization, since we have now the entire list:

$$(x - 2)(x - 3)(x + 7)(x + 4).$$

EXERCISES

Find the factors for

1. $x^4 + 4x^3 - 19x^2 - 46x + 120$.
2. $x^4 - 2x^3 - 23x^2 - 12x + 36$.

3. $x^4 - 11x^3 + 38x^2 - 52x + 24$.
4. $x^5 + x^4 - 13x^3 - x^2 + 48x - 36$.
5. $x^4 - 8x^3 + 24x^2 - 32x + 16$.
6. $x^4 + 10x^3 - 80x^2 - 480x + 2304$.
7. $2x^4 - 9x^3 + 4x^2 + 21x - 18$.
8. $3x^4 + 2x^3 - 33x^2 - 8x + 84$.
9. $7x^4 - 33x^3 + 23x^2 + 57x - 54$.
10. $3x^5 + x^4 - 35x^3 - 5x^2 + 72x - 36$.

3. We find of course in the process of trying out different divisors that many trial divisors give remainders. These may be utilized to cut down the list of possible divisors by a process we shall call *sifting*. It depends upon the following obvious theorem. Suppose an expression we are trying to factor does have a factor $x - a$, then the expression can be written $(x - a)\ Q$, where Q is the other factor, the quotient, if we were to divide by $x - a$. If now in the original expression, which we may call F, we substitute any number whatever for x, and get as result the number R, then if we make the same substitution in the form $(x - a)Q$, we must get the same number R as the product of the two numbers coming from $(x - a)$ and Q. That is, so far as our purpose is concerned, the number from $x - a$ must be a divisor of the number we have from F. Now when we have made a division and get a remainder, this remainder, as we have seen, is the value of F for the number a in the divisor. Hence if we have a table of possible divisors and go down this table substituting a for x in each one, the number resulting must be a divisor of the remainder we obtained, or the supposed factor is really not a factor at all. An example will make this clear.

Consider $x^3 - 9x^2 - 22x + 240$. The possible divisors are in the table: $x \pm 1$, $x \pm 2$, $x \pm 3$, $x \pm 4$, $x \pm 5$,

$x \pm 6$, $x \pm 8$, $x \pm 10$, etc. We will not write the entire list as we may not need it. We now try division by $x - 1$.

$$\begin{array}{r} 1 - 9 - 22 + 240 \,(1 \\ \underline{\quad 1 - \ 8 - \ 30} \\ 1 - 8 - 30, \quad 210 \end{array}$$

The remainder, 210, shows that $x - 1$ is not a factor. Substituting 1 for x in the table, we have the numbers 2, 3, 1, 4, 2, 5, 3, 6, 4, 7, 5, 9, 7, 11, 9, $\cdots$. Now 210 is not divisible by 4, 9, or 11, hence we cross out $x - 1$, $x + 3$, $x - 5$, $x + 8$, $x + 10$, $x - 10$. The new list is $x + 1$, $x \pm 2$, $x - 3$, $x \pm 4$, $x + 5$, $x \pm 6$, $x - 8$, $\cdots$. We try next 2.

$$\begin{array}{r} 1 - 9 - 22 + 240 \,(2 \\ \underline{\quad 2 - 14 - \ 72} \\ 168 \end{array}$$

We now substitute 2 for x in the reduced list of possible factors, getting

$$3, 4, -1, 6, 2, 7, 8, 4, 6.$$

These all divide 168, so there is no reduction. We now try 5.

$$\begin{array}{r} 1 - 9 - 22 + 240 \,(5 \\ \underline{\quad 5 - 20 - 210} \\ 30 \end{array}$$

Substituting 5 for x in the table we have 6, 7, 3, 2, 9, 1, 10, 11, 1, 3. Crossing out the expressions corresponding to 7, 9, 11, we have left the table

$$x + 1, x - 3, x - 4, x + 5, x - 6, x - 8.$$

We try next $x - 4$.

$$\begin{array}{r} 1 - 9 - 22 + 240 \,(4 \\ \underline{\quad 4 - 20 - 168} \\ 72 \end{array}$$

Substituting 4 we cross out $x + 1$, $x - 4$. We try now $x - 8$, and find it divides, then $x - 6$, then $x + 5$. The

best arrangement is shown in the following example, where the * marks the expressions to be omitted.

$x^4 - 4x^3 - 35x^2 + 78x + 360$ (1
$1 \quad - 3 \quad - 38 \quad 40$
400

$1 - 4 - 35 + 78 + 360$ (3
$3 - 3 - 114 - 108$
252

$1 - 4 - 35 + 78 + 360$ (−1
$-1 + 5 + 30 - 108$
252

$1 - 4 - 35 + 78 + 360$ (−2
$-2 + 12 + 46 - 248$
112

$1 - 4 - 35 + 78 + 360$ (6
$6 + 12 - 138 - 360$

$1 + 2 - 23 - 60$ (5
$5 + 35 + 60$

$1 + 7 + 12$ (−4
$-4 - 12$

$1 + 3$

$(x + 3)(x + 4)(x - 5)(x - 6)$.
Factors.

	1	3	−1	−2
$(252)x + 1$	2	4	*	
$(400)x - 1$	*			
$(112)x + 2$	3*			
$x - 2$	1	1	3	4
$x + 3$	4	6	2	1
$(252)x - 3$	2	*		
$x + 4$	5	7	3	2
$x - 4$	3*			
$x + 5$	6*			
$x - 5$	4	2	6	7
$x + 6$	7*			
$x - 6$	5	3	7	8
$x + 8$	9*			
$x - 8$	7*			
$x + 9$	10	12	8*	
$x - 9$	8	6	10*	
$x + 10$	11*			
$x - 10$	9*			

It should be noted that five expressions were still in the list, but if enough substitutions were made, the actual list left would all be factors, and only these would be factors. It might, however, take a good many substitutions.

4. We will notice finally the polynomials that start with other coefficients than 1. For the same reason that the possible factors have to end with numbers that are factors of the final number of the polynomial, so they must start with coefficients of x which will be factors of the

coefficient of the highest power of x. For instance, the entire table of possible divisors of $6x^3-19x^2-26x+24$ is

$x \pm 1$ $x \pm 2$ $x \pm 3$ $x \pm 4$ $x \pm 6$ $x \pm 8$ $x \pm 12$ $x \pm 24$
$2x \pm 1$ $2x \pm 3$ $3x \pm 1$ $3x \pm 2$ $3x \pm 4$ $3x \pm 8$ $6x \pm 1$.

In order to sift this list we substitute $x = 1$.

$$\begin{array}{r} 6 - 19 - 26 + 24 \ (1 \\ 6 - 13 - 39 \\ \hline -15 \end{array}$$

Putting $x = 1$ in the table, we have left $x \pm 2$, $x \pm 4$, $x - 6$, $2x \pm 1$, $2x \pm 3$, $3x \pm 2$, $3x - 4$, $3x - 8$, $6x - 1$. Substituting $x = 2$, we have -56, and from the table we have left $x + 2$, $x - 4$, $x - 6$, $2x \pm 3$, $3x \pm 2$, $3x - 4$, $3x - 8$. We next substitute 3, which gives -63, and left in the table are $x - 4$, $x - 6$, $2x \pm 3$, $3x - 2$, $3x - 8$. In order to try the factor $2x + 3$, we may write it $2(x + \frac{3}{2})$ and then try $x + \frac{3}{2}$.

$$\begin{array}{r} 6 - 19 - 26 + 24 \ (- \frac{3}{2} \\ - 9 + 42 - 24 \\ \hline 6 - 28 + 16 \end{array}$$

The factors are then $(6x^2 - 28x + 16)(x + \frac{3}{2})$. We may transfer 2 from the first factor to the second, so we have $(3x^2 - 14x + 8)(2x + 3)$. Further factoring gives the three:

$$(x - 4)(3x - 2)(2x + 3).$$

The student should understand that polynomials of this type need not have any linear factors whose coefficients are whole numbers. A factor of the form $ax + b$, where a and b are integers, is called a *rational linear factor*. *The rational factors of polynomials may be of higher degree than first.*

EXERCISES

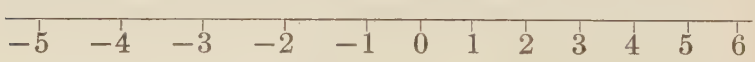
Factor the following for the rational factors.

1. $2x^3 - 11x^2 + 17x - 6$.
2. $2x^3 - x^2 - 7x + 6$.
3. $3x^3 - 2x^2 - 7x - 2$.
4. $6x^3 - 17x^2 - 4x + 3$.
5. $6x^3 - x^2 - 11x + 6$.
6. $12x^3 + 20x^2 - 37x - 30$.
7. $12x^3 - 4x^2 - 25x + 12$.
8. $2x^3 + x^2 + 8x - 21$.
9. $3x^3 - 4x^2 + 11x - 10$.
10. $40x^3 - 2x^2 - 69x + 36$.

5. In order to study expressions that do not have rational factors we must introduce a notion that originated with Descartes. This brilliant idea connected algebra and geometry, and in fact made such a synthesis of the two that after his time it has been seen by all mathematicians that geometry and algebra are in a sense the same subject looked at from two different standpoints.

What we do is to make a *graph* of an expression. This is a curve drawn in the following manner. We assign to x, the unknown letter in the expression, a series of values, as, $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots$, finding the value of the expression for each such value of x. This value of the expression we will represent by y.

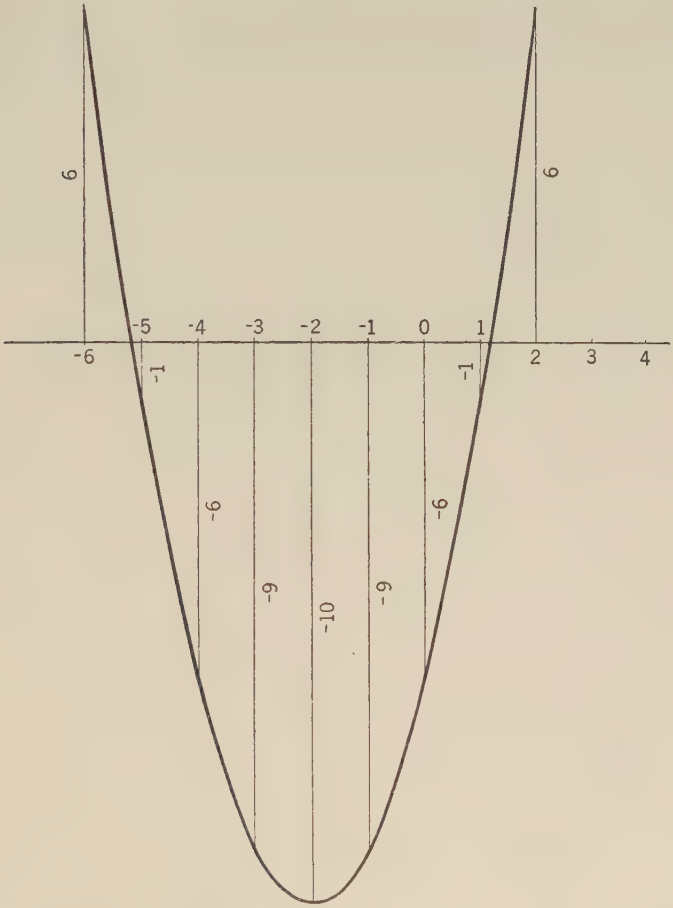
We now draw a straight line on which we lay off the values of x on a convenient scale, starting from some point we mark 0. The positive values are laid off to the right, usually, and the negative values to the left. Thus:



We now erect a perpendicular at each value of x, which will be the length of the value y for that value of x. For instance, for the expression $x^2 + 4x - 6$, we have the values

$x =$	-6	-5	-4	-3	-2	-1	0	1	2	3	4
$y =$	6	-1	-6	-9	-10	-9	-6	-1	6	15	26

These values we now represent on the diagram below, drawing the perpendicular for y up when y is positive and down when it is negative. The scale of y does not



have to be the same as that for x. The values are found only to that value of x, beyond which it is evident that they will get larger and larger if we proceed farther. After the ordinates for y have been drawn, we pass as

smooth a curve as we can through their end-points. This curve is (when placed on the scale) called a *graph*. The curve above is the graph of the expression $x^2 + 4x - 6$. If we needed to make a more accurate curve, we could calculate the values of y for intermediate values of x. For instance, for $x = -5.2$, $y = +.24$; for $x = 1.162$, $y = -.001756$.

The values of x for which the curve crosses the line drawn for x, called the axis of x, are called *roots* of the expression. Each root corresponds to a factor. Of course the roots may be whole numbers, and in the case of rational factors would be either integers or else exact fractions. If there are no rational factors, or roots, the values of the roots could not be represented by decimal fractions nor indeed by any other fractions. In the example above the roots have to be expressed in terms of the square root of 10. The method of doing this is considered later.

If the graph is carefully drawn, we can estimate a value of x which will give a very small value for y. Such a value of x is called an approximate root. For instance, in the case above an approximate root is 1.162, and corresponding to it would be the factor $x - 1.162$. Another approximate root would be -5.162, corresponding to the factor $x + 5.162$. The quadratic would then be approximately the product of the factors

$$(x - 1.162)(x + 5.162).$$

A closer approximation would be given by

$$(x - 1.1622777)(x + 5.1622777),$$

as the student may see by multiplying these together. The method of deriving these approximate values we will take up immediately.

An expression like $ax + b$ will have a straight line for its graph.

EXERCISES

Make graphs for the following expressions.

1. $x^2 - 2x - 3$.
2. $x^2 + 2x - 3$.
3. $x^2 - 2x + 3$.
4. $x^2 + 2x + 3$.
5. $x^2 - 3x - 10$.
6. $x^2 + 4x - 5$.
7. $x^2 + 4x + 4$.
8. $x^2 - 6x + 9$.
9. $x^2 + 3x - 6$.
10. $x^2 - 4x + 2$.
11. $x^3 - 6x^2 + 11x - 6$.
12. $x^3 - 2x - 8$.
13. $x^3 - 3x^2 - 6x + 8$.
14. $x^3 + 2x^2 - 4x - 8$.
15. $x^3 + x^2 - 3x + 2$.

In each case where the graph crosses the axis of x, estimate the approximate value.

6. We consider next an important process of changing an expression, called *transforming*. As an instance let us express the following in powers of $(x - 2)$ in place of powers of x, $x^3 + 3x^2 - 5x - 15$. To do this we merely have to divide by $x - 2$ three times, thus

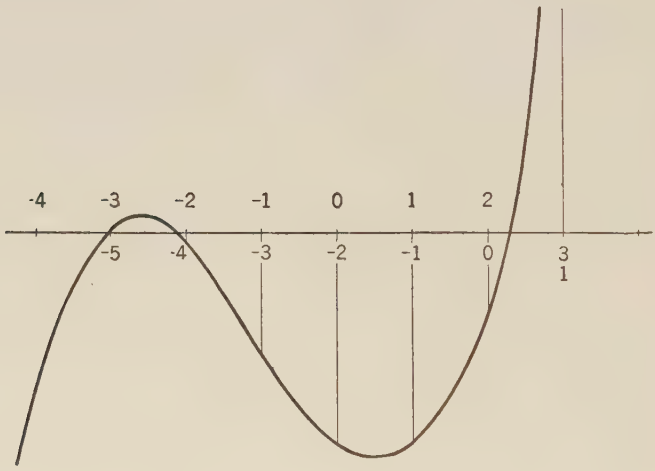
1	+ 3	− 5	− 15	(2
	2	10	10	
1	5	5	− 5	
	2	14		
1	7	19		
	2			
1,	9			

This means, since a dividend is always the product of the divisor by the quotient plus the remainder, that

$$x^3 + 3x^2 - 5x - 15 = (x - 2)(x^2 + 5x + 5) - 5$$
$$= (x - 2)(x - 2)[x + 7] + (x - 2)19 - 5$$
$$= (x - 2)(x - 2)(x - 2) + (x - 2)(x - 2)9 + (x - 2)19 - 5,$$

or finally $\quad 1(x - 2)^3 + 9(x - 2)^2 + 19(x - 2) - 5.$

This process is entirely similar to that on page 11 where a number represented on the scale of 10 is changed to a representation on the scale of 7. It means that if we renumber our x-scale so that what was 2 is now 0, what was 3 is now 1, what was -1 is now -3, etc., then the graph for the transformed cubic plotted on the new scale is the same curve as that of the original expression on the original scale.



This method of transforming enables us to move the zero-point as close as we please to a root. In the above there is a root between 2 and 3, and by transforming to 2, we have a new equation with the same curve as its graph. In this new equation the value at 0 is -5. Also the distance from the new zero (the old 2) is now only a few tenths. But the cube of tenths is thousandths, and the square is hundredths; consequently if we leave off the first two terms of the expression, we shall not be far from the value zero with those left. That is, we write

$$19(x - 2) - 5 \text{ is near zero.}$$

We can now approximate to the value of the tenths as $+\frac{5}{19} = +0.2^{+}$. We may now transform again to the new .2 (or the old 2.2), thus

$1 + 9.0$	$+ 19.00$	$- 5.000$	(.2
2	1.84	4.168	
1 92	20 84	$-$ .832	
2	1 88		
1 94	22 72		
2			
$1 + 9.6$	$+ 22.72$	$-$ .832	

We can see also that if we had transformed to .3, we would have gone past the crossing, for we would have

1 90	.1900	$-$ 5000	(3
3	279	6537	
1 93	2179	$+$ 1537	

and the + sign of the remainder shows that the value of the expression has changed sign, and the curve is now above the axis, whereas it should be below.

Repeating the process, if we divide + .832 by 22.72, we get .036, as being an approximation to the value of the root, which would be written 2.236.

In making these approximations, the first estimate of the tenths is often not close, and the student must learn to see that what is finally settled on is the largest number that can be used and not give the wrong sign. For instance,

$$x^3 - 3x^2 - 4x + 8$$

has a root between 3 and 4. Transforming to 3, we have

$$(x-3)^3 + 6(x-3)^2 + 5(x-3) - 4.$$

If now we divide 4 by 5, we get .8. Guessing, however, that this is too large, since the coefficient of x^2 is 6 and will furnish a good deal to carry to the 5, we try .7. This

makes the last term (the remainder) 2.783, which is +, hence .6 should be tried. This makes the remainder 1.376, so we try .5, which gives .125, still too large, and we have the tenths figure finally, .4. This gives as the transformed expression

$$(x - 3.4)^3 + 7.2(x - 3.4)^2 + 10.28(x - 3.4) - .976.$$

We can now, however, estimate the hundredths exactly if we stop with two decimals, as it comes out .09. If we wish to go farther, however, we would find this a trifle large. The value of the root to four figures is 3.4893.

The decimal point is usually omitted, as the ciphers annexed give the proper columns for the figures. Care must be exercised also when a 0 occurs in the figures of the root. In general, after m figures have been obtained by carrying on the process in full, $m - 1$ may be obtained by the division instead of a single one.

EXERCISES

In each case a graph should be drawn to locate the roots, which should then be computed to two decimal places.

1. $x^3 - 6x^2 + 3x + 11.$
2. $x^3 - 5x^2 - 2x + 8.$
3. $x^3 + 4x^2 - 5x - 16.$
4. $x^3 - 4x^2 - 4x + 8.$
5. $x^4 + 4x^3 - x^2 - 10x + 4.$
6. $x^2 + 3x - 6.$
7. $x^2 - 4x - 10.$
8. $x^2 + 6x + 3.$
9. $x^2 + 8x + 8.$
10. $x^2 - 6x + 7.$
11. $x^3 - 3x^2 - 6x + 12 = 0.$
12. $x^3 - x^2 - 7x + 9 = 0.$
13. $x^3 - 2x^2 - 7x + 10 = 0.$
14. $x^3 - 4x^2 - x + 11 = 0.$
15. $x^3 - 3x^2 - 2x + 9 = 0.$

The process explained above is one of several for calculating the approximate value of roots. It is due to Horner.

7. Another method can be exemplified by the quadratic.

For instance, consider $x^2 + 3x - 6 = 0$. If we transpose and divide by x, we have

$$x = -3 + \frac{6}{x}.$$

In this we substitute for x on the right the expression x on the left equals, giving

$$x = -3 + \cfrac{6}{-3 + \cfrac{6}{x}}.$$

We can continue this process, which for convenience will be written on a line,

$$x = -3 \cfrac{6}{+ -3 + \cfrac{6}{-3 + \cfrac{6}{-3 + \cfrac{6}{-3 + \cfrac{6}{-3}}}}} \cdots.$$

Such an expression is called a *continued fraction.* If we stop at any point, the terms left out would of course affect the value of what we should get, but we will at least have an approximate value for x. Thus above we would have

$$\frac{6}{-3} = -2, \quad -3 - 2 = -5, \quad \frac{6}{-5} = -\frac{6}{5}, \quad -3 - \frac{6}{5} = -\frac{21}{5},$$

$$\frac{6}{-\frac{21}{5}} = -\frac{10}{7}, \quad -3 - \frac{10}{7} = -\frac{31}{7}, \quad \frac{6}{-\frac{31}{7}} = -\frac{42}{31},$$

$$-3 - \frac{42}{31} = -\frac{135}{31}, \quad \frac{6}{-\frac{135}{31}} = -\frac{62}{45}, \quad -3 - \frac{62}{45} = -4.38\cdots.$$

The value computed by Horner's method is -4.38.

The student will notice the rhythm in the expression above, shown in the repetition of the fractions. It can be shown that any repeating continued fraction gives the rhythmic expression for the irrational roots of a quadratic if the continued fraction has meaning. Suppose the continued fraction is

$$x = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \frac{a_n}{b_n} + \frac{a_1}{b_1} + \cdots.$$

Then

$$x = \frac{a_1}{b_1 +} \cdots \frac{a_n}{b_n + x}.$$

When this is reduced it gives a quadratic equation for x. For instance, let the fraction be

$$x = \frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{1}{3} \cdots.$$

Then we have $$x = \frac{1}{3} + \frac{2}{5} + \frac{3}{7 + x} = \frac{38 + 5x}{128 + 17x}.$$

Whence x is a root of the quadratic

$$17x^2 + 123x - 38 = 0.$$

As an instance of a quadratic whose roots cannot be written this way consider $x^2 - x + 1 = 0$. This gives

$$x = 1 - \frac{1}{x} = 1 - \frac{1}{1} - \frac{1}{1} - \frac{1}{1} - \frac{1}{1} \cdots.$$

This fraction has no meaning, for if we attempt to find its value at any stopping-point, we strike at once the meaningless fraction $\frac{1}{1 - 1}$.

8. We saw in the first paragraph of this chapter that -420, the last number of the expression $x^4 - 11x^3 + 11x^2 + 179x - 420$, is the product of the numbers 3, 5, 7, -4, that is, is the product of the roots. It is not difficult to see that the second coefficient, -11, is the sum of these numbers with the sign changed. If we ask what relation there is between these roots and the other coefficients, 11 and 179, it is evident that the first, 11, is the sum of the products of the roots two at a time. This would give $15 + 21 - 12 + 35 - 20 - 28 = 11$, and the other, -179, is the sum of the products three at a time, with its sign changed. We have

$$105 - 60 - 140 - 84 = -179.$$

In the general case for any four roots α, β, γ, δ, we would have

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2 - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + \alpha\beta\gamma\delta.$$

If we write this in the shortened form

$$x^4 + p_1x^3 + p_2x^2 + p_3x + p_4,$$

where p stands for product, and the subscript shows how many roots are used to make a product, and it is understood that p also implies the sum of all such products, then we have by comparing the two forms of the expression:

sum of the roots one at a time

$$= (\alpha+\beta+\gamma+\delta) = -p_1,$$

sum of the products of pairs

$$= (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) = p_2,$$

sum of the products of triples

$$= (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta) = -p_3,$$

sum of the products in fours

$$= \alpha\beta\gamma\delta = p_4.$$

These expressions in terms of the roots are *symmetric*, that is, it does not matter which root is called α, or β, or γ, or δ, for if we interchange these letters in each of the expressions, we have precisely the same expressions, only in a different order. These expressions are called the *elementary symmetric expressions for the roots* α, β, γ, δ. It is evident that if we consider any other equation, we would have similar expressions as the coefficients, whatever the degree of the equation. For instance, in $x^3 - 4x^2 + 5x + 17 = 0$ the sum of the three roots is 4, the sum of the products of the three pairs is 5, the product of the three is -17.

9. We will now study the quadratic from this point of view, deriving all the information we can from the expressions we get this way. Let the quadratic be

$$ax^2 + bx + c = 0.$$

Then, since to have forms similar to those above we would have to divide first by a, we have, α and β being the two roots,

$$\alpha + \beta = -\frac{b}{a},$$

$$\alpha\beta = \frac{c}{a}.$$

That is, in any quadratic we can read the sum and the products of the roots directly out of the equation, without knowing the roots themselves. For instance, in

$$3x^2 + 13x - 10 = 0$$

the sum of the roots is $-\frac{13}{3}$ and the product is $-\frac{10}{3}$. (The student may factor and find the roots and verify that these numbers are correct.)

It would occur to us naturally that from the two equations

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}$$

we may by determinants eliminate either α or β, and then solve for the other. However, we find that we arrive at the original quadratic, so that we have gone around in a circle. We may, however, still reach a solution by a method which will introduce some new and important notions.

There is a theorem (which we shall not undertake to prove in an elementary book) that any rational symmetric function can be expressed rationally in terms of the elementary symmetric functions. If then we consider $\alpha - \beta$, which is not symmetric, since if we interchange α and β

we have $\beta - \alpha$, which is not equal to $\alpha - \beta$; but which is such that if we square it, $(\alpha - \beta)^2 = \alpha^2 - 2\,\alpha\beta + \beta^2$; we have a symmetric expression, we may make progress in the solving. In fact, we find with no trouble that by squaring the sum $\alpha + \beta$, we have

$$\alpha^2 + 2\,\alpha\beta + \beta^2 = \frac{b^2}{a^2},$$

and since

$$4\,\alpha\beta = \frac{4\,c}{a},$$

we have, after reducing to the common denominator a^2 and subtracting,

$$\alpha^2 - 2\,\alpha\beta + \beta^2 = \frac{b^2 - 4\,ac}{a^2} = \frac{\Delta}{a^2}.$$

The number Δ, which stands for $b^2 - 4\,ac$, is called the *discriminant* of the quadratic. We can now see that, by taking the square root,

$$\alpha - \beta = \frac{\sqrt{\Delta}}{a},$$

we have taken only the plus sign for the radical, which means α is the greater root. With this new value we are in a better position to find α and β. For now if we add

$$\alpha + \beta = -\frac{b}{a},$$

$$\alpha - \beta = \frac{\sqrt{\Delta}}{a}$$

and divide by 2, we have

$$\alpha = \frac{-\,b + \sqrt{\Delta}}{2\,a}.$$

If we subtract and divide by 2, we have

$$\beta = \frac{-\,b - \sqrt{\Delta}}{2\,a}.$$

This solves the quadratic completely for all cases. It is

called the *formula* solution for the quadratic. For instance, if we have

$$3x^2 - 8x - 12 = 0,$$

$$a = 3, b = -8, c = -12, \Delta = 208 = 16 \cdot 13, \sqrt{\Delta} = 4\sqrt{13}.$$

Hence

$$\alpha = \frac{8 + 4\sqrt{13}}{6} = \frac{4}{3} + \frac{2\sqrt{13}}{3},$$

$$\beta = \frac{4}{3} - \frac{2\sqrt{13}}{3}.$$

Of course if Δ is a perfect square, the roots will be rational and could be found by factoring. For instance,

$$6x^2 - x - 35,$$

$$a = 6, \quad b = -1, \quad c = -35, \quad \Delta = 841, \quad \sqrt{\Delta} = 29.$$

$$\alpha = \tfrac{5}{2}, \beta = -\tfrac{7}{3}.$$

10. The method of solving an equation by finding some further unsymmetric expression in terms of the roots is due to Lagrange. The expression chosen must be one which can be set in an equation of a solvable degree, and which will give for the coefficients of the equation of which it is the unknown number, symmetric expressions in terms of the roots of the original equation. For instance here, if we put $\alpha - \beta = z$, then $z^2 = \alpha^2 - 2\alpha\beta + \beta^2$, which is symmetric and $= \dfrac{b^2 - 4ac}{a^2}$. In the case of a cubic the expressions are not so simple. Following Lagrange, and well versed in the work of Lagrange, was Galois. He undertook to find what is necessary in order to write the roots of the general equation. In this case the answer would be: we must know $\sqrt{\Delta}$. If this is rational, so are the roots. If this is irrational, it is a *quadratic irrationality*, that is, is a square root of a number which is not the square of some rational number. The first quadratic irrationality discovered was $\sqrt{2}$. This is

involved in expressing the diagonals of a square in terms of the sides. It was known to Pythagoras that there was no rational relation, or, as it was then expressed, the diagonal of a square is incommensurable with the side. Numbers which are the roots of equations with integral coefficients, whether rational or algebraic, are called *algebraic* numbers. Other irrationals are called *transcendental.* A well-known example of the latter kind is the number π. Late in the last century this was proved to be not the root of an equation with integers or algebraic numbers for its coefficients. This gave a definitive answer to the long-standing problem of *squaring the circle.*

11. Galois found that for every algebraic equation there were related equations called *resolvent* equations, such that if any root of a resolvent equation is known the others are all rationally expressible in terms of the one known, and the formulae of expressing the various roots in terms of the one known do not change for the different roots taken as the one known. The roots of the original algebraic equation are rationally expressible in terms of the given root of the Galois resolvent.

For instance, for the quadratic, the Galois resolvent commonly used is $z^2 = \Delta$. If the two roots of this are z_1 and z_2, we have $z_1 = -z_2$. The roots of the original quadratic are then

$$\frac{-b+z_1}{2a}, \quad \frac{-b-z_1}{2a}.$$

This is not the only resolvent, for the quadratic itself is a resolvent equation since we have

$$\alpha = -\frac{b}{a} - \beta.$$

However, all the roots of all the resolvent equations going with a given equation may be expressed rationally in terms of any one root of any one of the resolvents.

The roots of resolvent equations need not be expressible in terms of radicals, whether square roots, cube roots, or any other. For instance, the roots of the quintic (fifth degree) usually cannot be expressed by radicals. One of the problems Galois solved was that of finding when an equation could be solved in radicals. It was known some years before his time that the quintic, and presumably higher equations, could not be solved by any method known at that time.

Returning to the quadratic we may now list the important formulae as follows:

$$ax^2 + bx + c = 0$$

$$a = \qquad\qquad \alpha\beta = \frac{c}{a}$$

$$b = \qquad\qquad \alpha + \beta = -\frac{b}{a}$$

$$c = \qquad\qquad \alpha - \beta = \frac{\sqrt{\Delta}}{a}$$

$$\Delta =$$

$$\alpha = \frac{-b + \sqrt{\Delta}}{2a} \qquad \beta = \frac{-b - \sqrt{\Delta}}{2a}$$

EXERCISES

Write down all the formulae above for each quadratic, filling in the given data.

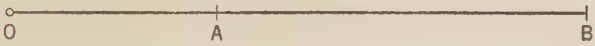
1. $x^2 + 9x + 18 = 0$.
2. $x^2 - 7x - 18 = 0$.
3. $x^2 - 11x - 30 = 0$.
4. $2x^2 + x - 21 = 0$.
5. $42x^2 + 19x - 35 = 0$.
6. $192x^2 + 244x - 3075 = 0$.
7. $x^2 + 13x + 21 = 0$.
8. $2x^2 - 12x + 11 = 0$.
9. $3x^2 + 15x - 29 = 0$.
10. $5x^2 + 24x - 17 = 0$.

12. There is another important feature to notice in the quadratic in certain cases. It may very well happen that Δ comes out a negative number. For instance, if we study $3x^2 - 8x + 7 = 0$, we find $\Delta = -20$.

The question then arises as to what the square root of a negative number is. The early mathematicians said there was no number such that its square would be negative. But in 1545 Cardan used the square root of a negative number in solving the equations, $x + y = 10$, $xy = 40$. He gave as the answers $5 + \sqrt{-15}, 5 - \sqrt{-15}$. Since that time the list of numbers actually used has included numbers whose square is negative. The original name of *imaginary number* has survived, but the student should know that these numbers are not imaginary any more than negative numbers are imaginary, or any other numbers. The notion of what we should call "number" has gradually become more comprehensive in the development of thought and today it includes a great deal more than it did even in Cardan's time. (See Smith's *History*, vol. 2, p. 261.)

13. To make the matter clear, suppose we define a number as the *ratio of two segments of lines*. That is, a number will tell us what to do to the first segment to make the second. We have many possible cases.

(1) Let the segments be on the same line in the same direction.



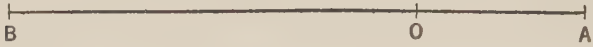
In the figure suppose we let a represent the ratio of OB to OA. We can write then

$$OB = a \cdot OA.$$

And to indicate that they have the same direction we will write the sign $+$ in front of the a; thus $OB = +a \cdot OA$. This, of course, gives a different meaning to the $+$ sign from that of addition. However, it produces no confusion. Logically it would have been better if in the history of mathematics this double use of $+$ and a corresponding double use of $-$ had never occurred. But the actual

difficulty is small. The equation just written means: change the length of OA by the multiplier a, then draw it in the same direction as OA, and we will have OB.

(2)

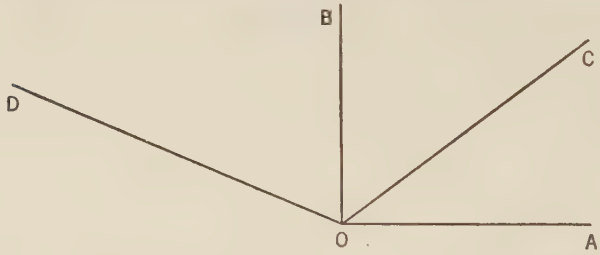


In the next place let the lines run in opposite directions. We will then write

$$OB = -\,a \cdot OA.$$

This gives us the directions: change OA in the ratio a, then draw it in the opposite direction. Such a number as $-\,a$ is then called a negative number, and the $-$ is used in a different sense from subtraction.

(3) If now we have the segments OA, OB neither in the same nor in opposite directions, as in the figure given,



we have other kinds of number. In this case if we wish to indicate the ratio of OB to OA, the angle between being 90°, we write $OB = a_{90} \cdot OA$. If OC is at an angle of 30°, we write $OC = c_{30} \cdot OA$. And if OD is at an angle of 150°, we write $OD = d_{150} \cdot OA$. It is evident that as the number of angles is unlimited, we cannot keep on introducing new signs like $+$ and $-$ indefinitely, and must use a subscript or other index notation.

If we consider the ratio of OD to OC, since the angle is 120° it will be some number f_{120}, so that $OD = f_{120}OC = f_{120}a_{30}OA$. We may then consider that the original

ratio $d_{150} = f_{120}a_{30}$. And as the numerical ratios would be multiplied, we may set down as our *definition of multiplication of these numbers*

$$f_{120}a_{30} = (af)_{150}.$$

In general we shall say that the product of two of these *signed* numbers will be a signed number, whose numerical part is the product of the numerical parts of the factors, and whose sign will be the sum of the sign-numbers of the factors. For instance,

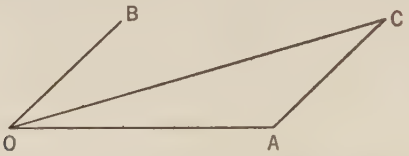
$$24_{20} \cdot 18_{36} = 432_{56}.$$

It is evident that a quotient can be found or defined in a reverse manner. For instance,

$$432_{56} \div 24_{20} = 18_{36}.$$

14. The numerical part of these numbers we may call its *arithmetic* part. The sign we shall call the algebraic part. We may say that the difference between algebra and arithmetic consists in the character of the numbers considered. Numbers in arithmetic are unsigned numbers. Those used in algebra are signed. When they have merely the signs + and − they are called *real*. If they have a single sign, as these we have just looked at, and this sign is neither 0, which corresponds to +, nor 180, which corresponds to −, we will call them *imaginary*, or *complex*, the term having lost its original sense, of fictitious, and referring now only to the sign. It is clear at once that if we draw segments out into space, there are other signed numbers, in such case needing two angles to locate the segment, and therefore two signs to identify the number. Such numbers are called *quaternions*. We do not consider them here. The usual notation for 1_{90} is i.

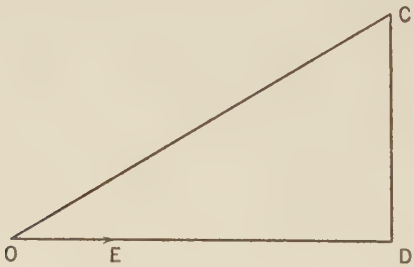
15. The question arises at once as to what we must call the sum and the difference of these signed numbers. The answer is given by a geometric addition and subtraction. We define the geometric sum of two directed segments thus: The sum of OA and OB is OC, found as follows. Draw OA, then from A draw AC parallel and equal to OB, then join O and C. OC is then defined to be the geometric sum of OA and OB, that is, $OA + OB = OC$.



We may now define the sum of two signed numbers as follows. If $OA = \alpha\, OX$, $OB = \beta\, OX$, and $OC = \gamma\, OX$, then we shall define

$$\alpha + \beta = \gamma.$$

It is immaterial what OX is. This definition enables us to see at once that if we construct on OC as a hypotenuse any right triangle,



$OD = a\, OE$, $DC = bi\, OE$, $OC = (a + bi)OE$. It is therefore possible to express all the different signed numbers of the plane by the use of $+$ and $-$, and i and $-i$. We would write here if $OC = r_\theta\, OE$,

$$r_\theta = a + bi.$$

The ratio of $\frac{a}{r}$ is called the *cosine of* θ, $\cos\theta$, and $\frac{b}{r}$ the *sine of* θ, $\sin\theta$. For different angles these numbers are given in tables, to four or more decimal places. For instance, if OC has a length 10 and its angle with OE is 26°, then

$$a = 8.9879 \quad \text{and} \quad b = 4.3837,$$
$$10_{26} = 8.9879 + 4.3837\,i.$$

The student should now prove that

$$a + bi + c + di = (a + c) + (b + d)i.$$

Since $i1$ is the ratio of a segment equal to another in length and perpendicular to it, we have at once

$$ii1 = -1.$$

Or, accommodating the use of exponents to these new numbers,

$$i^2 1 = -1.$$

We have here then an actual number whose square is negative unity. We generally write only the sign. We notice also

$$i^2 = -1,\ i^3 = -i,\ i^4 = +1,\ i^5 = i,\ i^6 = -1,\ i^7 = -i,$$
$$i^8 = +1, \cdots.$$

EXERCISES

Reduce the following.

1. $3 + 4\,i + 6 - 7\,i + 8 + 2\,i.$
2. $8 - 2\,i - (5 - 6\,i).$
3. $(4 + 3\,i)(4 - 3\,i).$
4. $(3 + 2\,i)(3 - 2\,i).$
5. $(5 + 2\,i)(5 - 2\,i).$

The student will notice in the last three problems that we have as results numbers which have so far been called prime. Now we see they are the products of signed numbers as factors, so they need not be considered as prime. Any prime of the form $4N + 1$ can be factored in this manner. We will call such numbers as $a + bi$ *complex*.

16. In order to make a division, let us write

$$a + bi = (x + yi)(c + di),$$

which is virtually a definition of what division means. What we ask is: What multiplier of $c + di$ will give $a + bi$? We have after multiplying together the factors on the right

$$cx - dy = a,$$
$$dx + cy = b.$$

Solving these by determinants,

$$x = \frac{ac + bd}{c^2 + d^2}, \quad y = \frac{bc - ad}{c^2 + d^2}.$$

If we write the problem in the usual manner for ordinary reals,

$$x + yi = \frac{a + bi}{c + di},$$

we may arrive at the same answer by multiplying the numerator and the denominator of the fraction by $c - di$. This gives at once

$$x + yi = \frac{ac + bd + (bc - ad)i}{c^2 + d^2}.$$

We can now return to our quadratic, which gave rise to all this consideration of complex numbers, and find that we can solve quadratics in which Δ is negative as easily as any others. For instance,

$$3x^2 + 5x + 7 = 0, \qquad \Delta = -59.$$
$$\alpha = \frac{-5 + i\sqrt{59}}{6}, \qquad \beta = \frac{-5 - i\sqrt{59}}{6}.$$

By substituting either of these values for x, we find the equation is satisfied. The formula for solving the quadratic is now seen to be perfectly general and complete. A question that would arise at once would be this: Since we now have a wider list of numbers, we will

have to consider that the coefficients themselves may belong to the general list or range of numbers; if then a quadratic has complex numbers for one or more of its coefficients, will the formula still work or must we add a further generalization of the range of numbers? The answer is that we have come to an end for the present, since the square root of a complex number is simply a complex number. For instance,

$$\sqrt{i} = \tfrac{1}{2}\sqrt{2} \pm \tfrac{1}{2}\sqrt{2}\,i.$$

This is easy to see, for squaring, we have: $\frac{1}{2} - \frac{1}{2} + i = i$.

In fact any algebraic equation whose coefficients are complex or real numbers will have as roots real or complex numbers. The complex range is then *closed*. There are other numbers than complex numbers, but they do not enter what is called elementary algebra.

17. If we return now to the quadratic, we may study not only the symmetric expressions in the roots, but also the symmetric expressions in the factors. For instance, for the quadratic $x^2 - 8x + 15$ we have the corresponding forms

Roots	Factors
$\alpha\beta = 15$	$(x-3)(x-5) = x^2 - 8x + 15 = Q$
$\alpha + \beta = 8$	$(x-3)+(x-5) = 2x - 8 = Q'$
$\Delta = 4$	$(\alpha - \beta)^2 = 4 = Q''$

The expressions in the factors are called Sturm's[1] functions. The roots of Q are 3 and 5; the root of Q' is 4, which, it is to be noticed, lies between 3 and 5. We notice further that when $x = 4$, the value of Q is -1, so that where $Q' = 0$, Q and Q'' have opposite signs. If we make a table of the signs of these functions for different values of x, we have

[1] Sturm was a Swiss mathematician. He discovered his theorem while studying the motion of compound pendulums.

	$x =$	0	1	2	3	4	5	6	7
	Q	+	+	+	0	−	0	+	+
	Q'	−	−	−	−	0	+	+	+
Changes	Q''	+	+	+	+	+	+	+	+
in sign		2	2	2	1	1	0	0	0

We notice there is a drop in the number of changes between the signs for $x = 2$ and $x = 4$: a drop of 1, and this is due to the change in sign of Q, and there is 1 root between 2 and 4. There is another drop between 4 and 6 of 1, due entirely to the change in sign in Q, and there is a root between 4 and 6. This is a general proposition applying to any quadratic. It is true because the changes in sign of Q' always occur where Q and Q'' have opposite signs. And either $-++$ and $--+$, or $-++$ and $--+$ or $++-$ and $+--$ or $+--$ and $++-$ give in each case the same number of changes. Hence changes in Q' do not affect the total number. Q'' does not change, hence the only drops will be due to the changes in Q. That is, there is a loss of one change for every root of Q passed. This is *Sturm's theorem* for the quadratic. It applies to any equation when the proper functions have been found. These functions may be found without knowing the factors, because, being symmetric, they depend only on the coefficients.[2]

18. In the general case

$$ax^2 + bx + c = Q,$$

we have

$$2ax + b = Q',$$
$$b^2 - 4ac = Q''.$$

We see that we can write down Q' by the rule that we multiply each coefficient by the exponent, and reduce the

[2] The parallelism between the symmetric expressions in the roots and in the factors should be noted. It is part of the fundamental design underlying the quadratic.

exponent by 1. This gives Q'. To get Q'' we divide Q by Q' and keep the remainder. This remainder with all signs changed is Q''. If we were dealing with a cubic or higher equation, we continue the division process, always changing the signs of the remainders. Also we note that in this case we should get

$$Q'' = \frac{b^2 - 4\,ac}{4},$$

but as we are interested only in the signs we may get rid of fractions by multiplying by the necessary positive numbers.

19. For a cubic we have[3]

$$\begin{aligned} Q &= ax^3 + bx^2 + cx + d, \\ Q' &= 3\,ax^2 + 2\,bx + c, \\ Q'' &= (2\,b^2 - 6\,ac)x + bc - 9\,ad, \\ Q''' &= 18\,abcd - 4\,ac^3 - 4\,b^3d + b^2c^2 - 27\,a^2d^2. \end{aligned}$$

We have the relations

$$\begin{aligned} 9\,aQ &= (3\,ax + b)Q' - Q'', \\ (2\,b^2 - 6\,ac)^2Q' &= Q''\,[\qquad\quad] - Q'''. \end{aligned}$$

These show that when Q' is zero, Q and Q'' have opposite signs, and when Q'' is zero, Q' and Q''' have opposite signs. That is to say, if we substitute a root of Q' or of Q'', there will be no variation in the number of changes of sign. The only difference will be when we pass a root of Q. For instance,

$$\begin{aligned} Q &= x^3 - 3\,x^2 - 12\,x + 8, \\ Q' &= 3\,x^2 - 6\,x - 12 \quad \text{or} \quad x^2 - 2\,x - 4, \\ Q'' &= 10\,x - 4 \quad \text{or} \quad 5\,x - 2, \\ Q''' &= +\,36. \end{aligned}$$

[3] In factors these are $Q = (x - \alpha)(x - \beta)(x - \gamma)$.

$$\begin{aligned} Q' &= [(x - \alpha)(x - \beta) + (x - \alpha)(x - \gamma) + (x - \beta)(x - \gamma)], \\ Q'' &= (\alpha - \beta)^2(x - \gamma) + (\alpha - \gamma)^2(x - \beta) + (\beta - \gamma)^2(x - \alpha), \\ Q''' &= (\alpha - \beta)^2(\alpha - \gamma)^2(\beta - \gamma)^2. \end{aligned}$$

If we make the table of signs, we have

$x=$	-10	-3	-2	0	2	4	5	6
Q	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$+$
Q'	$+$	$+$	$+$	$-$	$-$	$+$	$+$	$+$
Q''	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$
Q'''	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
var	3	3	2	2	1	1	1	0

These numbers show that the equation has one root between -3 and -2, one between 0 and 2, and one between 5 and 6. The theorem of Sturm is especially useful in problems that have roots very close together. For instance, the equation $x^3 - 3x^2 - 4x + 13 = 0$ shows on the graph the value $+1$ for $x = 2$ and also for $x = 3$. To find out whether the curve crosses between 2 and 3 we have the Sturm functions

$$x^3 - 3x^2 - 4x + 13,$$
$$3x^2 - 6x - 4,$$
$$2x - 5,$$
$$+1.$$

For 2 there are two changes of signs, for 3 there are none. Hence there are two roots between 2 and 3.

EXERCISES

1. $x^3 - x^2 - 7x + 9 = 0$.
2. $x^3 - 3x^2 - 12x + 26 = 0$.
3. $x^3 - 3x^2 - 12x + 20 = 0$.
4. $x^3 - 4x^2 - x + 11 = 0$.
5. $x^3 - 3x^2 - 6x + 12 = 0$.
6. $x^3 - 7x + 6 = 0$.
7. $x^4 + 2x^3 - 7x^2 - 8x + 13 = 0$.
8. $x^4 - 2x^3 - 31x^2 + 32x + 243 = 0$.
9. $x^3 + 2x^2 + 8x + 5 = 0$.
10. $x^3 - 2x^2 + 8x - 6 = 0$.
11. $x^3 + 3x^2 + 9x - 5 = 0$.
12. $x^3 - 6x^2 + 11x - 4 = 0$.
13. $x^4 + x^2 - 3x + 8 = 0$.
14. $x^4 + 2x^3 - 4x^2 - 6x + 13 = 0$.
15. $x^4 + x^2 + 1 = 0$.
16. $x^6 - 1 = 0$.

17. Find the radical to two decimal places:

$$\sqrt{23} \qquad \sqrt[3]{169} \qquad \sqrt[5]{285} \qquad \sqrt[6]{1279}$$

18. Show by Sturm's theorem that there are no real roots for $ax^2 + bx + c = 0$ when $b^2 - 4ac$ is negative.

19. Show by Sturm's theorem that there are only two real roots for $x^6 - 8 = 0$.

20. Apply Sturm's theorem to $x^2 - 8x + 16 = 0$.

20. We may finish the problem of solving two simultaneous equations. For instance, let the equations be

$$x + 2y = 5,$$
$$x^2 - xy = 6.$$

The determinant we get by eliminating y is

$$\begin{vmatrix} 2 & x - 5 \\ -x & x^2 - 6 \end{vmatrix} = 0.$$

This reduces to $3x^2 - 5x - 12 = 0$. From this

$$x = \frac{5 \pm \sqrt{169}}{6} = 3 \text{ or } -\tfrac{4}{3}.$$

If we had the equations

$$x + 2y = 5,$$
$$x^2 - xy = 8,$$

we would have had

$$3x^2 - 5x - 16 = 0,$$
$$x = \frac{5 \pm \sqrt{221}}{6}.$$

If we start with the equations

$$x^2 + 2y = 5,$$
$$x + 3y^2 = 12,$$

the determinant reduces to $3x^4 - 30x^2 + 4x + 27 = 0$. This has 2 positive roots, one between 1 and 2, the other between 5 and 6. The student should find them, and see if there are negative roots.

EXERCISES

1.	$x + y - 5 = 0$	$x^4 + y^4 - 97 = 0.$
2.	$x^2 + 3xy - 54$	$xy + 4y^2 - 115 = 0.$
3.	$x + y + 1 = 0$	$x^5 + y^5 + 211 = 0.$
4.	$x^2 + y = 11$	$x + y^2 = 7.$
5.	$y^2 - 8x - 16 = 0$	$x^2 + 3y - 2 = 0.$
6.	$x^3 - y = 4$	$x^2 + y^2 - 25 = 0.$
7.	$yx^2 + y = 3x$	$x^2 - 5y = 4.$

21. In order to solve the cubic, we consider first the form

$$x^3 - 3px + q = 0.$$

We need for the solution to find a cubic form which we can factor, and we might choose $A^3 + B^3$. However, it will be more convenient to select the form

$$A^3 - 3ABC + B^3 + C^3.$$

The factors of this are

$$A + B + C, \quad A + B\omega + C\omega^2, \quad A + B\omega^2 + C\omega,$$

where $\omega^3 = 1$ and $\omega = -\frac{1}{2} + i\frac{1}{2}\sqrt{3}$.

If we try to consider the cubic as having this form, it means that we take x as the A,

$$p = BC, \quad q = B^3 + C^3.$$

If we can find B and C from these equations, we will be able to write the cubic in a form we can factor. This is easy if we write the equations as

$$q = B^3 + C^3,$$
$$p^3 = B^3C^3.$$

The solution leads to the sixth-degree equation

$$Z^6 - qZ^3 + p^3 = 0.$$

As this has the form of a quadratic, we have first

$$Z^3 = \frac{q \pm \sqrt{q^2 - 4p^3}}{2}.$$

One of the signs will furnish B^3, the other C^3. When we take the cube root, we remember the cube roots of M will be $\sqrt[3]{M}$, $\omega\sqrt[3]{M}$, $\omega^2\sqrt[3]{M}$, and so the six roots will be the numbers we need for the six terms following a in the factors. As an example let the cubic be $x^3 - 6x - 4 = 0$. Here we have $p = 2$, $q = -4$. The sextic is

$$Z^6 + 4Z^3 + 8 = 0,$$
$$Z^3 = -2 + 2i \quad \text{or} \quad -2 - 2i,$$
$$B = 1 + i, \quad C = 1 - i.$$

Hence the cubic could be written

$$x^3 - 3x(1 + i)(1 - i) + (1 + i)^3 + (1 - i)^3 = 0.$$

The factors are, therefore,

$$(x+1+i+1-i)(x+\omega+\omega i+\omega^2-\omega^2 i)(x+\omega^2+\omega^2 i+\omega-\omega i)$$
$$= (x + 2)(x - 1 + i\sqrt{3})(x - 1 - i\sqrt{3}).$$

This is the solution of the cubic called *Cardan's*. It is not really due to Cardan but was first published by him in his Ars Magna (1545). The difficulty is in getting the cube root of complex numbers, which usually would have to be done with the aid of tables from trigonometry.

22. We will conclude the chapter on theory of equations with a further study of the ideas of Galois. First for the quadratic. Let the quadratic be written

$$x^2 + px + q = 0.$$

The roots are α and β, so that

$$\alpha + \beta = -p, \quad \alpha\beta = q, \quad \Delta = p^2 - 4q.$$

We saw that we could solve this by adjoining the equation $\alpha - \beta = \sqrt{\Delta}$. Suppose we now make a new equation whose roots will be

$$u\alpha + v\beta \quad \text{and} \quad u\beta + v\alpha,$$

where the numbers u, v are to be any that we choose. In the expression $\alpha - \beta$, $u = 1$, $v = -1$. These roots

are made by interchanging α and β. The new equation will also be a quadratic.

$$(z - u\alpha - v\beta)(z - u\beta - v\alpha) =$$
$$z^2 - (u + v)(\alpha + \beta)z + \alpha\beta(u^2 + v^2) + uv(\alpha^2 + \beta^2) = 0.$$

Since

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\,\alpha\beta=p^2-2\,q, \quad \text{and} \quad \alpha+\beta=-p, \quad \alpha\beta=q,$$

we have finally

$$z^2 + p(u + v)z + uvp^2 + (u - v)^2q = 0.$$

In the first place we will notice that the discriminant for the new equation is

$$p^2(u + v)^2 - 4\,uvp^2 - 4(u - v)^2q = (u - v)^2(p^2 - 4\,q)$$
$$= (u - v)^2\Delta.$$

Since the roots depend upon the square root of the discriminant, we see at once that the new equation will depend upon $\sqrt{\Delta}$ just as does the original. If the original has rational roots, so will the new equation, unless u and v are irrational. That is, the new equation has the irrationalities of the original and of the numbers u and v. As a particular case let us choose $u = 1$, $v = -1$, then we have as above

$$z^2 - p^2 + 4\,q = 0, \text{ giving } z = \sqrt{\Delta} \quad \text{or}$$
$$\alpha - \beta = \sqrt{\Delta}.$$

Since $\alpha + \beta = -p$, we can recover the formula for solving the quadratic.

The general equation in u and v is called the *general Galois resolvent.*

If we choose $u = 1 + i$, $v = 1 - i$, we have

$$(z + p)^2 - (-\Delta) = 0 \quad \text{and}$$
$$z = -p + i\sqrt{-\Delta} \quad \text{or}$$
$$(\alpha + \beta) + i(\alpha - \beta) = -p + i\sqrt{-\Delta}.$$

23. In this case we have introduced the irrational i through the coefficients u, v. In case the roots are rational, it will disappear on account of the $-\Delta$. But any irrationality produced by Δ itself will not disappear. It is a *natural* irrationality. The chief contribution of Galois to the solution of equations was his proof that any equation depends for the expression of its roots upon a single irrationality, which will be given by a general Galois resolvent equation or any special form. This single irrationality may be expressible in terms of simpler irrational numbers, but they will all come from the auxiliary resolvent equations used in solving the general equation. For instance,

$$x^4 + 4x^3 - 4x^2 - 16x - 8 = 0$$

depends for its solution on the resolvent equation $x^4 - 10x^2 + 1 = 0$, whose roots are $\sqrt{3}+\sqrt{2}$, $\sqrt{3}-\sqrt{2}$, $-\sqrt{3}+\sqrt{2}$, $-\sqrt{3}-\sqrt{2}$. It can, however, be solved in terms of the irrationals $\sqrt{3}$ and $\sqrt{2}$. This will appear if we write $x^4 - 10x^2 + 1 = (x^2-1)^2 - 8x^2 = (x^2 + 2\sqrt{2}x - 1)(x^2 - 2\sqrt{2}x - 1)$. The irrational $\rho = \sqrt{3}+\sqrt{2}$ is a single irrational, however. Indeed, we can express $2\sqrt{3} = \rho + \rho^{-1}$, $2\sqrt{2} = \rho - \rho^{-1}$. The roots of

$$x^4 - 4x^3 - 190x^2 + 388x - 191 = 0$$

are

$-1 + \frac{9}{2}\rho - \frac{1}{2}\rho^{-1}$		$-1 + 4\sqrt{3} + 5\sqrt{2}$
$-1 - \frac{1}{2}\rho + \frac{9}{2}\rho^{-1}$	that is also	$-1 + 4\sqrt{3} - 5\sqrt{2}$
$-1 + \frac{1}{2}\rho - \frac{9}{2}\rho^{-1}$		$-1 - 4\sqrt{3} + 5\sqrt{2}$
$-1 - \frac{9}{2}\rho + \frac{1}{2}\rho^{-1}$		$-1 - 4\sqrt{3} - 5\sqrt{2}$

The natural irrationalities are ρ, or $\sqrt{3}$ and $\sqrt{2}$.

24. In the Cardan solution of the cubic, the irrationality ω is not a natural one, and, in fact, the solution fails to be useful for the case of three real roots, since it brings in the cube root of two complex numbers and ω, when

the solution needs to be in real terms only. No effective solution of this kind is known for the general cubic. In the Cardan solution six roots for a general resolvent are formed: $\alpha + \beta\omega + \gamma\omega^2$, $\beta + \gamma\omega + \alpha\omega^2$, $\gamma + \alpha\omega + \beta\omega^2$, $\alpha + \beta\omega^2 + \gamma\omega$, $\beta + \gamma\omega^2 + \alpha\omega$, and $\gamma + \alpha\omega^2 + \beta\omega$. If we work out the resolvent for three numbers u, v, w, we have to find the equation whose roots are

$$\begin{aligned} &u\alpha + v\beta + w\gamma, \\ &u\beta + v\gamma + w\alpha, \\ &u\gamma + v\alpha + w\beta, \\ &u\alpha + v\gamma + w\beta, \\ &u\gamma + v\beta + w\alpha, \\ &u\beta + v\alpha + w\gamma, \end{aligned}$$

and while the reduction is a little tedious, it involves no particular trouble. In the first place, $ax^3+3\,bx^2+3\,cx+d$ may be written (after multiplying by a^2)

$$z^3 + 3\,Hz + G, \qquad \text{where } z = ax + b, \qquad H = ac - b^2,$$
$$G = a^2d - 3\,abc + 2\,b^3.$$

Let us write the cubic whose roots are u, v, w, and reduce it to the form

$$t^3 + 3\,H't + G'.$$

Also let $a^2\Delta = G^2 + 4\,H^3$, $a'^2\Delta' = G'^2 + 4\,H'^3$.

Let the unknown number for the sextic with the six given roots be ϕ, and let $\phi_0 = aa'\phi - 3\,bb'$. Then we will have

$$[\phi_0{}^3 - 27\,HH'\phi_0 - \tfrac{27}{2}\,GG']^2 - \tfrac{729}{4}(G^2+4\,H^3)(G'^2+4\,H'^3) = 0.$$

This is the *general Galois resolvent* for the cubic. When $H' = 0$, we revert to the Cardan solution, which means we have practically chosen u, v, w to be 1, ω, ω^2. For other choices the resolvent is no simpler than the original equation. The natural irrationality to enter first is the square root of $G^2 + 4\,H^3$. If this is a square, the

resolvent breaks up into two cubic equations with rational coefficients. If it is not a square, we will have two cubics whose last terms depend on $\sqrt{\Delta}$ or $-\sqrt{\Delta}$. Further irrationalities will depend upon the cubics. If we make $G' = 0$ by a proper choice of u, v, w (as $2w = u + v$), and put $\phi_0 = 3\zeta$, we will have, after making $H' = 1$, as we can,

$$[\zeta^3 - 3H\zeta]^2 - (G^2 + 4H^3) = 0.$$

This gives two cubics like the original, except for the last terms,

$$\zeta^3 - 3H\zeta \pm \sqrt{\Delta} = 0.$$

The irrationalities of the cubic then will depend naturally on $\sqrt{\Delta}$ and whatever other cubic irrationality arises from the first two terms of the cubic, set equal to a number involving only the $\sqrt{\Delta}$. If instead of making $H' = 1$ we set it equal to $-\frac{1}{3}$, which is equivalent to taking $u = -1$, $v = 1$, $w = 0$, and which gives the roots of the sextic as the differences of the roots of the cubic, $-\alpha + \beta$, $\alpha - \beta$, $-\beta + \gamma$, $\beta - \gamma$, $-\gamma + \alpha$, $\gamma - \alpha$, we will have

$$(\zeta^3 - H\zeta)^2 + \frac{\Delta}{27} = 0.$$

This is a simple Galois resolvent. For instance, let the cubic be

$$x^3 - 6x + 4 = 0,\ G = 4,\ H = -2,\ \Delta = -16.$$

We have as resolvent

$$(\zeta^3 - 2\zeta)^2 - \tfrac{16}{27} = 0,$$

or the two equations

$$\zeta^3 - 2\zeta \pm 4\frac{\sqrt{3}}{9} = 0.$$

A little trial shows that a root of these may be written

$$\pm \frac{2\sqrt{3}}{}$$

the upper signs corresponding, and the lower corresponding. The other roots are, for the upper sign,

$$\pm 1 - \frac{\sqrt{3}}{3},$$

and for the lower signs,

$$\pm 1 + \frac{\sqrt{3}}{3}.$$

We see from this the roots are all expressible rationally in terms of $\sqrt{3}$, or $\sqrt{\Delta}$. If the cubic had been

$$x^3 - 6x - 9, G = -9, H = -2, \Delta = 49,$$

we would have $\zeta^3 - 2\zeta \pm \frac{7\sqrt{3}}{9}$.

If we construct the Δ for this equation, or pair of equations, we find it is $+$; hence there are imaginary roots. So we try $\pm \frac{i}{3}\sqrt{3}$, and we find it goes, and all the roots are then expressible in terms of $i\sqrt{3}$. If we consider the cubic

$$x^3 - 7x + 6 = 0, G = 6, H = -\tfrac{7}{3}, \Delta = -\tfrac{400}{27},$$

we find

$$\zeta^3 - \tfrac{7}{3}\zeta \pm \tfrac{20}{27} = 0.$$

Since no radical is involved in this, the sextic has no square roots involved. If we write $3\zeta = y$, this becomes

$$y^3 - 21y \pm 20 = 0.$$

The roots of this are expressible in terms of any one of them rationally, and as only cube roots can enter, it becomes evident that the equation must have rational roots, as also does the first. The Cardan solution of the first would have given the irrational forms $\sqrt[3]{(3 + i\,10)} + \sqrt[3]{(3 - i\,10)}$, $\omega\sqrt[3]{(3 + i\,10)} + \omega^2\sqrt[3]{(3 - i\,10)}$, etc.

CHAPTER VI

IRRATIONAL NUMBERS

1. We have arrived now at a point from which we can see that in addition to rational numbers we must also have others, which we have called *irrational.* It was shown, as stated, by Galois, that for every algebraic equation there is a resolvent equation, the same resolvent belonging to an endless number of other equations, such that the resolvent equation determines a single irrational number. In terms of this irrational number, all the roots of the resolvent equation may be stated in rational form, and all the roots of the original equation. If then we were in possession of all the resolvent, or *Galois Normal Equations,* we should have from each one a new irrational number defined, and from all of them the endless totality of all algebraic irrationals. Such numbers are called *algebraic numbers,* although this term includes also all rational numbers.

2. Many of these numbers have simple representations, in symbols, as the solutions of the binomial equations

$$x^n - a = 0.$$

One of the roots of this equation is $\sqrt[n]{a}$. All radicals are of this form, though they are also represented with fractions for exponents, thus

$$\sqrt[n]{a} = a^{\frac{1}{n}}.$$

Solutions of other equations, as, for instance, $x^3 - 3x - 1 = 0$, are not usually represented in any special manner.

We might introduce in a particular case some symbol to represent the number, as here we might use u; then we would have for the powers of u: u, u^2, $u^3 = 3u + 1$, $u^4 = 3u^2 + u$, $u^5 = 3u^3 + u^2 = u^2 + 9u + 3$, etc. Any polynomial in u could be reduced to a quadratic in u.

3. The different roots of a resolvent equation are called *conjugate* to one another. For instance, in the equation $x^2 = 7$, the two conjugate roots are $+\sqrt{7}$ and $-\sqrt{7}$. The symmetric functions of the conjugate roots are rational numbers.

4. There is a general theorem to the effect that if an equation has algebraic numbers for its coefficients, its roots are algebraic numbers. The domain of algebraic numbers is thus seen to be a *closed* domain for algebraic operations. To arrive at other kinds of numbers, we would have to introduce some new processes. The simplest algebraic irrationals are those expressible by radicals. These we will study a little more closely.

5. Thus far most of our expressions have been free from radicals, and were for the most part rational. We must now make some study of expressions which have radicals. In order to do this we must go back to the meaning of an exponent.

When we multiply x by x we write x^2 in place of $x \cdot x$. So too x^3 means $x \cdot x \cdot x$. It is easy then to see that if a and c are integers,

$$x^a x^c = x^{a+c}.$$

We make use of this fact in order to define the meaning of any exponent which is a rational number. That is, we lay down as a definition that whatever rational numbers a and c are, $x^a x^c = x^{a+c}$. We have then the following consequences ($x \neq 0$).

I. $x^a x^0 = x^a$, from which we find that $x^0 = 1$, whatever x.

II. $x^c x^{-c} = x^0 = 1$. Therefore, $x^{-c} = \frac{1}{x^c}$. Hence a negative exponent means "invert" the number it is attached to.

III. $\underbrace{x^{\frac{1}{a}} x^{\frac{1}{a}} x^{\frac{1}{a}} x^{\frac{1}{a}} \cdots x^{\frac{1}{a}}}_{a \text{ factors}} = x^{\frac{a}{a}} = x$ when a is an integer and positive. From this it is necessary that

$$x^{\frac{1}{a}} \quad \text{means} \quad \sqrt[a]{x}.$$

IV. Since $x^{\frac{1}{a}} x^{\frac{1}{a}} x^{\frac{1}{a}} \cdots x^{\frac{1}{a}}$ (c factors) $= x^{\frac{c}{a}} = (\sqrt[a]{x^c})$, it follows that a fractional exponent means a power given by the numerator of the root indicated by the denominator. We can now interpret the meaning of any rational exponent. For instance,

$$64^{\frac{2}{3}} = (\sqrt[3]{64})^2 = 4^2 = 16, \qquad 64^{-2} = \tfrac{1}{4096},$$
$$125^{-\frac{4}{3}} = \tfrac{1}{625}, \qquad (\tfrac{128}{2187})^{-\frac{3}{7}} = \tfrac{27}{8}.$$

Fractional exponents are more convenient than radical signs, and usually they are to be preferred. Negative exponents are used in elaborate expressions, but can be dispensed with in ordinary expressions. To get rid of them we usually multiply the numerator and denominator of the fraction by some number that will cancel all the negative exponents. An example is

$$\frac{x^{-2} + y^{-2}}{x^{-3} + y^{-3}}.$$

We multiply above and below by x^3y^3. This gives, leaving out the letters with zero exponent as they $= 1$,

$$\frac{xy^3 + x^3y}{y^3 + x^3} = \frac{xy(x^2 + y^2)}{(x + y)(x^2 - xy + y^2)}.$$

If there are no denominators, we collect the terms after

removing the numbers with the highest negative exponents. For instance,

$$3x^{-5} + 4x^{-2} - yx + 8x^3 = x^{-5}(3 + 4x^3 - yx^6 + 8x^8),$$
$$4x(1 - x^2)^{-\frac{3}{2}} + 6(1 - x^2)^{-\frac{1}{2}} = (1 - x^2)^{-\frac{3}{2}}[4x + 6(1 - x^2)].$$

It is always desirable to collect the irrational expression under a single fractional exponent. An example in handling numbers is the following:

$$\begin{aligned}\sqrt{50} - \sqrt{28} + 3\sqrt{32} - 5\sqrt{98} &= \sqrt{(25 \cdot 2)} - \sqrt{(4 \cdot 7)} + 3\sqrt{(16 \cdot 2)} - 5\sqrt{(49 \cdot 2)} \\ &= 5\sqrt{2} - 2\sqrt{7} + 12\sqrt{2} - 35\sqrt{2} \\ &= -2\sqrt{7} - 18\sqrt{2}.\end{aligned}$$

Since a radical is the root of an equation of degree corresponding to the index of the root, the other roots of the equation will play a part in handling problems containing the radical. The other roots are called *conjugate* roots, and they involve the conjugate roots of the Galois resolvent belonging to the radical equation. For instance, the number $\sqrt{3}$ is the root of $x^2 - 3 = 0$. The other root is $-\sqrt{3}$. If we have such a fraction as

$$\frac{5 + \sqrt{3}}{2 - \sqrt{3}},$$

we can get the radical into either the numerator or the denominator alone. To remove it from the denominator, we multiply above and below by the conjugate root $2 + \sqrt{3}$. This gives

$$\frac{5 + \sqrt{3}}{2 - \sqrt{3}} \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{13 + 7\sqrt{3}}{1} = 13 + 7\sqrt{3}.$$

To remove the radical from the numerator, we multiply by the conjugate root of $5 + \sqrt{3}$, which is $5 - \sqrt{3}$. This gives

$$\frac{22}{13 - 7\sqrt{3}}.$$

Similarly, we reduce

$$\sqrt{\frac{1+x}{1-x}} \text{ to } \frac{\sqrt{(1-x^2)}}{1-x} \text{ or to } \frac{1+x}{\sqrt{(1-x^2)}}.$$

6. Equations are treated in the same way. To solve

$$\sqrt{(x+4)}+\sqrt{(2x+1)}=9$$

we separate the radicals, giving $\sqrt{(2x+1)}=9-\sqrt{(x+4)}$; then we multiply each side by itself, giving

$$2x+1=85+x-18\sqrt{(x+4)}.$$

Transposing everything but the radical,

$$x-84=-18\sqrt{(x+4)}.$$

Multiplying each side by itself, we have

$$x^2-168x+7056=324x+1296.$$

This gives us the rational equation

$$x^2-492x+5760=0,\ x=12 \text{ or } 480.$$

Substituting 12, we find we have $\sqrt{16}+\sqrt{25}=9$, which is true for the radicals taken as positive numbers. Substituting 480, we have $\sqrt{484}+\sqrt{961}=9$, which is true only if the first radical is taken negatively and the second positively. We have evidently solved two different equations in this process. To see where the extra equation came in let us go back and treat the problem entirely with conjugate radicals. To do this we transpose the 9.

$$\sqrt{(x+4)}+\sqrt{(2x+1)}-9=0.$$

We multiply now by the conjugates

$$-\sqrt{(x+4)}+\sqrt{(2x+1)}-9,\ \sqrt{(x+4)}-\sqrt{(2x+1)}-9,$$
$$\text{and}\quad -\sqrt{(x+4)}-\sqrt{(2x+1)}-9.$$

This gives us the same quadratic as before, but we now see it as the product of four conjugate expressions, each

of which might be zero. The last multiplier would require the sum of two negative numbers to be positive, which is impossible. The next to the last multiplier would require $\sqrt{(x+4)}$ to be greater than $\sqrt{(2x+1)}$, or $x+4$ greater than $2x+1$, which would require x to be less than 3. As this will not be possible, two of the introduced factors have no solution. The root 480 belongs to the first multiplier, and the root 12 to the original equation.

Equations which have exactly the same roots are called *equivalent*. Each is then reducible to the other. Likewise a system of equations are equivalent when they have exactly the same roots.

EXERCISES

1. Reduce each of the following: $36^{\frac{1}{2}}$, $27^{\frac{1}{3}}$, $16^{\frac{1}{4}}$, $125^{\frac{2}{3}}$, $729^{\frac{5}{6}}$, $64^{.5}$, $81^{.75}$.

2. Reduce $(x^{\frac{2}{3}})^{\frac{3}{4}}$, $(x^{-\frac{2}{3}})^{-\frac{6}{5}}$, $x^2x^3x^{-6}$, $x^{\frac{1}{2}}x^{\frac{1}{4}}x^{\frac{2}{3}}$.

3. Reduce $\sqrt{12}$, $\sqrt{50}$, $\sqrt{72}$, $\sqrt[3]{256}$, $\sqrt[3]{729}$, $\sqrt[3]{343}$, $\sqrt[4]{81}$, $\sqrt[4]{96}$.

4. $\sqrt{x^2y^3z^4}$, $\sqrt{x^5y^3z^6}$, $3\sqrt{x^9y^4z^6}$, $\sqrt[3]{x^5y^2z^7}$.

5. Reduce $\sqrt{(a^3+2a^2b)} - \sqrt{(a^3-2a^2b+ab^2)} - \sqrt{4ab^2}$.

6. Rationalize the denominators:

$$\frac{5}{3+\sqrt{5}}, \quad \frac{7+a}{4-\sqrt{a}}, \quad \frac{6-\sqrt{5}}{3\sqrt{5}-4}.$$

7. Rationalize the numerators:

$$\frac{\sqrt{(1+x^2)}}{\sqrt{(1-x^2)}}, \quad \frac{\sqrt{(2ax-x^2)}}{\sqrt{(a-x)}}, \quad \frac{\sqrt[3]{(1-x^4)}}{\sqrt[3]{(1+x^2)}}.$$

8. Rationalize the denominators:

$$\frac{6-7i}{2+3i}, \quad \frac{8+9i}{4-3i}, \quad \frac{-3+5i}{3+5i}, \quad \frac{4-6i}{1+i}, \quad i=\sqrt{-1}.$$

9. Solve where possible:

$$\sqrt{(x+4)}+\sqrt{(x-3)}=7,$$
$$\sqrt{(x-5)}-\sqrt{(x+4)}-1=0,$$
$$\sqrt{(3x+1)}-(7x-7)=-2.$$

10. Solve the quadratic by formula, then substitute the roots and prove they are correct:

$$3x^2-7x+19=0,$$
$$4x^2+5x+3=0,$$
$$2x^2+9x-23=0.$$

LOGARITHMS

7. There is one extension of the meaning of number beyond algebraic numbers which we must notice. It arises as follows. If we recall the laws of exponents,

$$A^x \cdot A^z = A^{x+z},$$
$$A^x : A^z = A^{x-z},$$
$$(A^x)^z = A^{xz},$$

we see that in case we can represent numbers as powers of A, we can turn the process of multiplication into a process of adding, division into subtracting, and raising to a power to multiplying. When numbers are represented as powers of a base A, the exponents of the powers are called *logarithms on the base A*. The base usually chosen is the base of the decimal system of writing numbers, that is, 10. Logarithms are, for the most part, *transcendental*, that is, *not algebraic numbers*.

The idea of using logarithms was first effectively developed by Lord Napier, about 1600, though he did not use the base ten. This was introduced by Briggs, a professor at Oxford, shortly afterward. Tables of logarithms to three, four, five, six, seven, ten, and even further decimal places have been computed. The student should use the tables for the calculations.

8. In using the tables it must be remembered that the powers of ten itself will have integers for exponents; thus, $10^4 = 10{,}000$, $10^3 = 1000$, $10^2 = 100$, $10^1 = 10$, $10^0 = 1$, $10^{-1} = .1$, $10^{-2} = .01 \cdots$.

Since a number such as 232 is 2.32×100, if we have the exponent for 2.32, the exponent for 232 would be found by adding 2 to the exponent of 2.32. This would be (using the tables) $2.32 = 10^{.3655}$, $100 = 10^{2.0000}$; hence

$$(2.32)(100) = 10^{.3655}10^{2.0000} = 10^{2.3655}.$$

It was due to Briggs that this feature of logarithms on the base ten was noticed. We make a simple rule out of it as follows: To find the logarithm of a number on the base ten, look up the decimal (called the *mantissa*) in the logarithm table for the original number written with a decimal point after its first figure. Prefix to this mantissa the *characteristic* (see Chapter II) of the number. For instance, we have

$2 = 10^{.3010}$	$20 = 10^{1.3010}$	$2000 = 10^{3.3010}$
$3 = 10^{.4771}$	$30 = 10^{1.4771}$	$30{,}000 = 10^{4.4771}$
$5 = 10^{.6990}$	$500 = 10^{2.6990}$	$5{,}000{,}000 = 10^{6.6990}$
$1.8 = 10^{.2553}$	$18 = 10^{1.2553}$	$1800 = 10^{3.2553}$
$7.46 = 10^{.8727}$	$74.6 = 10^{1.8727}$	$746 = 10^{2.8727}$
$.746 = 10^{.8727-1}$	$.00746 = 10^{.8727-3}$	

We will now write a few examples in the use of logarithms, showing what is done. In the first case we will write the work in full, then in the usual abbreviated manner.

a. Find the product $(245)(.721)(3.96)(1800)(.0025)$. Substituting for each its proper form as a power of 10, we have

$$(10^{2.3892})(10^{.8579-1})(10^{.5977})(10^{3.2553})(10^{.3979-3}) = 10^{3.4980}.$$

The result is between 3140 and 3150. The proper exponent for 3140 is 3.4969 and for 3150 is 3.4983. There is a difference of 14, and as our number is 3.4980, which is

11 over the lowest of the two, the number we are trying to find must be $\frac{11}{14}$, or .8, of the way from 3140 to 3150; that is, it is close to 3148. This is the answer to the problem to four figures. The student should realize that nearly all calculations with logarithms are approximate. The usual manner of writing this calculation is as follows:

$$\begin{aligned}
&\log \ 245 = 2.3892\\
&\log \ .721 = \ .8579 - 1\\
&\log \ 3.96 = \ .5977\\
&\log \ 1800 = 3.2553\\
&\log .0025 = \ .3979 - 3\\
&\log \text{ ans.} = 3.4980
\end{aligned}$$

Ans. 3148.

b. To find the logarithms of numbers of four figures we insert proportional values just as we did above. For instance, to get the logarithm of 32.67. This is .7 of the way between 32.60 and 32.70. The two logarithms of these numbers are 1.5132 and 1.5145. The difference is 13 and $\frac{7}{10}$ of this is 9. This makes

$$\log 32.67 = 1.5132 + 0009 = 1.5141.$$

As an instance of division we take the problem

c. How many liters of hydrochloric acid gas would there be in 150 grams? The arithmetic statement is

$$x = \frac{150}{(.0896)(36.6)}.$$

The calculation is written

$$\begin{aligned}
&\log 36.6 = 1.5635\\
&\log .0896 = \ .9523 - 2\\
&\log \text{ denom.} = .5158
\end{aligned}$$

$$\begin{aligned}
&\log 150 = 2.1761\\
&\log \text{ denom.} = \ .5158\\
&\log x = 1.6603 \qquad x = 45.74.
\end{aligned}$$

characteristics not in the table
the table has only the mantissas

FOUR-PLACE LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

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FOUR-PLACE LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

d. In extracting a root we use the third law of exponents $(A^x)^z = A^{xz}$. For instance,

$$\sqrt[5]{(2.36)^3} = (2.36)^{\frac{3}{5}}$$

$$\log 2.36 = .3729$$

$$3$$

$$5)\overline{1.1187}$$

$$\log \text{ans.} = .2237 \quad \text{ans.} = 1.674.$$

e. We need a little care when the characteristic is negative. For instance,

$$(.0002734)^{\frac{3}{7}}$$

$$\log .0002734 = .4368 - 4$$

$$3$$

$$1.3104 - 12$$

As 12 would give a decimal when divided by 7, we change it to 14 and the characteristic 1 in front of the mantissa, to 3. Then we proceed

$$7)\overline{3.3104 - 14}$$

$.4729 - 2,$ which is the logarithm of .02971.

Sometimes the negative characteristics are always made to be -10 by changing the positive characteristic. For instance, the student may see

$$.2645 - 3 \text{ written } 7.2645 - 10.$$

This is usually the case in problems from trigonometry. There is no great advantage in it, however.

EXERCISES

1. The volume of a sphere is given by the formula

$$V = \frac{4\pi r^3}{3}.$$

For a radius of 4 inches find the volume of a sphere of iron.

2. A cubic inch of water weighs .03617 pounds. Iron weighs 7.5 as much as the same volume of water. Find the weight of the sphere in problem 1.

3. A liter of CO_2 weighs 44 times as much as a liter of hydrogen, which weighs .0896 grams at 0° temperature and 60 mm. pressure. How many liters of CO_2 will there be in 1000 grams?

4. If the temperature rises to 25° C., the volume will change by the factor $\frac{298}{273}$, and if the pressure drops to 745 mm., the volume will further increase by the factor $\frac{760}{745}$. Find the new volume of the kilogram of CO_2.

5. What would be the radius of a ball of iron as above which would weigh 100 grams?

6. The weight which it will take to crush a wrought iron-column is given by the formula

$$299{,}600\,\frac{d^{3.55}}{l^2}.$$

If a column is 12 feet long and has a diameter of 4 inches, what will be necessary to crush it?

7. The formula for the amount of water flowing over a weir is given as

$$3.33\,LH^{\frac{3}{2}}.$$

Find the value of this when L is 12 and H is .48.

8. If money is put at interest which is compounded semi-annually, at 3 per cent, 1 dollar will amount to 1.015 at the end of the first half-year. The value of 800 dollars in 16 years would then be $800(1.015)^{32}$. Calculate what this is.

area of a triangle $A = \sqrt{S(S-a)(S-b)(S-c)}$

$a = 1.414$
$b = 1.674$
$c = 2.853$
$2S = 5.941$
$S = 2.971$

S = Semi-perimeter

$2.971 = S$
$-1.414 = a$
$1.557 = S-a$

log

2.971	0.4729
1.557	0.1923
1.297	0.1129
.118	0.0719 −1
2	0.8500 −2 1.0000
	.9250 −1

9248 9250 9253

8410 8420

.8414 area

$1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \cdots$ u_n $n(n+2) \equiv ?$

$2\cdot 5 + 4\cdot 7 + 6\cdot 9 + \cdots$ $2n(2n+3) \equiv ?$

$3\cdot 7 + 5\cdot 10 + 7\cdot 13 + \cdots$ $? \equiv ?$

CHAPTER VII

IDENTITIES

1. We have already had occasion frequently to transform expressions into other forms which were identical with the first ones, meaning that whatever values (in general) we might assign to the letters the two expressions would reduce to the same number. As an example, $(x - a)(x + a) = x^2 - a^2$, whatever the values of x and a. We will now make a more complete study of certain types of forms and identities that come from them. Identities are of high importance, for they enable us to transform expressions into manageable forms. They exemplify the artistic principle *unity*.

Finite Series

Let us return to the triangle of Pascal, and write it in a little different arrangement. What we wrote as lines before we will make the diagonals now. Thus

1	1	1	1	1	1	1	1	. . .
1	2	3	4	5	6	7	8	. . .
1	3	6	10	15	21	28	36	. . .
1	4	10	20	35	56	84	120	. . .
1	5	15	35	70	126	210	330	. . .
1	6	21	56	126	252	462	792	. . .
1	.	.	.	.	.	.	.	. . .

We noticed then that the numbers in any line (here a diagonal) could be found from those in the preceding line by adding the two that were on either side of the one wanted. For instance, here in the third line $15 = 10 + 5$, the numbers

being taken now just before and just above. This will always be true, as we can see if we put back the factorial writing of the binomial theorem.[1] In the instance given we would have

$$\frac{6!}{4!2!} = \frac{5!}{3!2!} + \frac{5!}{4!1!} = \frac{4 \cdot 5! + 2 \cdot 5!}{4!2!} = \frac{6 \cdot 5!}{4!2!} = \frac{6!}{4!2!}$$

In the general case we would have the three fractions

$$\frac{n!}{k!\,(n-k)!} + \frac{n!}{(k+1)!\,(n-k-1)!} = \frac{(n+1)!}{(k+1)!\,(n-k)!}$$

These formulae are very useful, for we can now fill out any term in the k place in any row, and we can also see how to write the sum of any column down to the k row, for the sum will be the number just to the right of the one with which we stop.

For instance, the sum of the k column down to the nth term is the number in the row n and column $k + 1$, which would have the form

$$\frac{(k+n-1)!}{(n-1)!\,k!}.$$

In numerical form it is easy to write these for any number by noting the number of rows above and the number of columns to the left, as, for instance, 462 has five rows above and six columns to the left, and

$$462 = \frac{11!}{6!\,5!}.$$

The bottom term in the k column and in the n row is

$$\frac{(k+n-2)!}{(k-1)!\,(n-1)!}.$$

We can say the same for the k row and n column. We

[1] See page 12.

may then fill out the table first written for n columns and k rows, thus

$$1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \cdots\ 1$$
$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ \cdots\ n$$
$$1\ 3\ 6\ 10\ 15\ 21\ 28\ 36\ 45\ \cdots\ \frac{(n+1)n}{1\cdot 2}$$
$$1\ 4\ 10\ 20\ 35\ 56\ 84\ 120\ 165\ \cdots\ \frac{(n+2)(n+1)n}{1\cdot 2\cdot 3}$$
$$1\ 5\ 15\ 35\ 70\ 126\ 210\ 330\ 495\ \cdots\ \frac{(n+3)(n+2)(n+1)n}{1\cdot 2\cdot 3\cdot 4}$$
$$1\ k\ \ \frac{(k+1)k}{1\cdot 2}\ \cdots\ \frac{(n+k-2)(n+k-3)\ \cdots\ n}{1\cdot 2\cdot 3\cdot 4\ \cdots\ k-1}$$

In each case the formula at the end gives the nth term of that row, and the sum of the terms in the row above to that column. If we multiply the third row through by 2 !, we can write it

$1\cdot 2+2\cdot 3+3\cdot 4+4\cdot 5+5\cdot 6+\ \cdots\ n(n+1)$, and multiplying the formula for the sum below it, we have

$$s_n=\tfrac{1}{3}n(n+1)(n+2).$$

Multiplying the fourth line by 3 ! and noticing that the formula for the end term is the product of three successive numbers, we have

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+4\cdot 5\cdot 6+\ \cdots,$$
$$+\,n(n+1)(n+2)=\tfrac{1}{4}n(n+1)(n+2)(n+3).$$

We can see that the same method will enable us to write in general, using an abbreviated notation,

$$\Sigma n(n+1)(n+2)(n+3)\ \cdots\ (n+k)$$
$$=\frac{1}{k+2}n(n+1)(n+2)(n+3)\ \cdots\ (n+k+1).$$

We will now work in the reverse direction, by finding the difference between two successive terms in a row.

From the way they are constructed and their properties, we see that the difference of the successive numbers in the k row are the numbers in the $k-1$ row. If we take the differences of the differences, we have the second differences, which will be the $k-2$ row, etc.

We may also state now that if we have compound expressions made up of sums of terms in the same columns multiplied by any numbers, we can find the sum of the series of such terms. Thus, suppose we multiply all the terms of the first line by a, the second by b, the third by c, and add the corresponding terms, that is, those in the same column. The new series would be (using the forms above with no fractions)

$$\Sigma[a + bn + cn(n+1)] = \Sigma[cn^2 + (b+c)n + a].$$

Since any quadratic can be reduced to this form, we can now find the sum of a series whose terms are quadratics in n. The sums of the different series would be in this case

$$an + \frac{bn(n+1)}{1 \cdot 2} + \frac{cn(n+1)(n+2)}{1 \cdot 2 \cdot 3}.$$

Reducing this to a polynomial, we would have

$$\frac{cn^3 + 3(b+c)n^2 + (6a + 3b + 2c)n}{6}.$$

We can reduce any polynomial to a series of terms of this form by the usual method of Horner division. Suppose, for instance, we have to find the sum of the series

$$\Sigma(n^2 - 2n + 4).$$

We first transform the polynomial

$$n^2 - 2n + 4.$$

We write this in terms of the forms a, bn, $cn(n+1)$. To

do this, we first divide by n, then the quotient by $n + 1$, and keep the remainders. That will give us

$$\begin{array}{r|l} -1 & 1 \quad -2, +4 \\ & \underline{\quad -1} \\ & 1, -3 \end{array}$$

so that we have $n^2 - 2n + 4 = 4 - 3n + n(n + 1)$. This may easily be verified. Applying our method we find from this the sum formula would be

$$4n - \frac{3n(n+1)}{2} + \frac{n(n+1)(n+2)}{3}$$
$$= \frac{24n - 9n(n+1) + 2n(n+1)(n+2)}{6}.$$

which reduces to

$$\frac{2n^3 - 3n^2 + 19n}{6}.$$

Giving n the values 1, 2, 3, 4, 5, this gives

$$3, \quad 7, \quad 14, \quad 26 \cdots$$

with differences 3, 4, 7, 12, and these are what we arrive at when we make $n = 1, 2, 3$ in the formula for the difference. The simplest method for reducing to an ordinary polynomial is simply division worked backwards. To simplify, we multiply the expression by 6, and then arrange the coefficients thus

$$\begin{array}{l|l} -1 & \\ -2 & \underline{\qquad\qquad} \\ & \underline{\qquad\qquad} \\ & 2 \quad -9 \end{array} \Big| \begin{array}{l} \\ \\ \underline{\quad 24} \\ \\ \end{array}$$

The expressions on the left were the divisors. Then starting at the bottom we have

$$\begin{array}{lll|l} & 2 & -3 & 19 \\ -1 & & -2 & \underline{\quad 5} \\ -2 & 2 & -5 & 24 \\ & & -4 & \\ & \underline{2} & \underline{-9} & \end{array}$$

Reversing what we do in division we say $-2 \times 2 = -4$, subtract from -9 is -5. Then -1×2 is -2, subtract from -5 is -3, -1×-5 is 5, subtract from 24 is 19. This gives the top line. We may omit the unnecessary writing and condense the process as follows:

$$\begin{array}{cccccc} 1 & -2, & 4 & \frac{1}{3} & \frac{-3}{2} & 4 \quad \frac{1}{6} \quad (2 - 9 + 24) \\ & -1 & & & & \\ \hline 1, & -3 & & & & \end{array} \qquad \begin{array}{ccc} 2 & -3 & 19 \\ & -2 & 5 \\ \hline 2 & -5 & 24 \\ & -4 & \\ \hline 2 & -9 & \end{array}$$

$$s_n = \frac{2n^3 - 3n^2 + 19n}{6}.$$

As another example we will consider the series whose terms are given by n^2. The work is set forth as follows:

$$\begin{array}{ccccc} -1 & 1 & +0 & +0 & \frac{1}{3} \quad -\frac{1}{2} \quad 0 \\ & & -1 & 0 & \\ \hline & 1 & -1 & 0 & \end{array} \qquad \begin{array}{ccc} 2 & -3 & 0 \\ \hline & 6 & \end{array}$$

$$\begin{array}{ccccc} & & & 2 & 3 \quad 1 \\ & & -1 & & -2 \quad -1 \\ \hline & & & 2 & 1 \quad 0 \\ & & -2 & & -4 \\ \hline & & & 2 & -3 \end{array}$$

$$s_n = \frac{2n^3 + 3n^2 + n}{6} = \tfrac{1}{6}n(n+1)(2n+1).$$

That is,

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

EXERCISES

Prove the following formulae, first by arriving at the sum from the form for the n-term, then arriving at the n-term as the difference of two successive forms for the sum.

1. $1 + 3 + 5 + \cdots (2n - 1) = n^2$. Show this also graphically, as did Pythagoras.

2. $2 + 4 + 6 + 8 + \cdots 2n = n(n + 1)$. Show this by a proper rectangle.

3. $1^3 + 2^3 + 3^3 + 4^3 + \cdots n^3 = \frac{1}{4} n^2(n+1)^2$.

4. $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \cdots n(n+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5)$.

5. $1 \cdot 6 + 2 \cdot 7 + 3 \cdot 8 + \cdots + n(n+5) = \frac{1}{3} n(n+1)(n+8)$.

6. $1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \cdots + n(n+3) = \frac{1}{3} n(n+1)(n+5)$.

7. $1 + 4 + 7 + \cdots + (3n-2) = \frac{1}{2} n(3n-1)$.

8. $a + (a+d) + (a+2d) + (a+3d) + \cdots + a + (n-1)d = an + \frac{1}{2} dn(n-1)$. This is an *arithmetic progression.*

9. Show that if the difference is ar^n, the original n-term was $\frac{ar^{n+1} - a}{r-1}$. The series a, ar, $ar^2 \cdots$ is a *geometric progression.*

10. $1^2 + 3^2 + 5^2 + \cdots (2n-1)^2 = \frac{1}{3} n(2n+1)(2n-1)$.

11. $1 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + \cdots = ?$

12. If $\Delta = ar^n + br^{n+h} + cr^{n+k} + dr^{n+l} + \cdots$,

$$\text{then } s_n = \frac{a(r^{n+1}-1) + b(r^{n+h+1}-1) + \cdots}{r-1}$$

$$= \frac{r\Delta - (a+b+c+\cdots)}{r-1}.$$

2. Let us now look at series with differences of the differences. Consider first the series below and its differences with their differences:

1	4	9	16	25	36	$\cdots$
	3	5	7	11		
		2	2	2	2	
			0	0	0	

If we start with the first diagonal 1

3

2

we can reconstruct each line, thus 1

3 5 7 9 11

2 2 2 2

then 1 4 9 16 25

3 5 7 9

2 2 2

$\frac{1+3(n-1)+\cdots(n-1)(n-2)}{1\cdot 2}$

$\text{Sum} = 1n + \frac{3n(n-1)}{1\cdot 2} + \frac{2n(n-1)(n-2)}{1\cdot 2\cdot 3} = \frac{n}{6}\left(2n^2 - 6n + 4 \;\; 9n - 9\right)$

$2n^2 + 3n + 6$

If, now, we apply this method to a general case where the first diagonal is

$$\begin{array}{ccccccc} a & & & & & & \\ & b & & & & & \\ & & c & & & & \\ & & & d & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{array}$$

we have

$$\begin{array}{ccccccc} a & a+b & & a+2b+c & & a+3b+3c+d & \cdots \\ & b & & b+c & & b+2c+d & \\ & & c & & c+d & & \\ & & & d & & & \end{array}$$

It is evident that in any case, if the lowest line of differences are all alike, say a_k, then the next line above will be

$$a_{k-1},\ a_{k-1}+a_k,\ \cdots a_{k-1}+ma_k,\ \cdots.$$

The line above that will be

$$a_{k-2}+a_{k-1},\ a_{k-2}+2\,a_{k-1}+a_k,$$
$$a_{k-2}+ma_{k-1}+\frac{m(m-1)}{2}a_k\cdots.$$

The next line

$$a_{k-3},\ a_{k-3}+a_{k-2},\ a_{k-3}+2\,a_{k-2}+a_{k-1},$$
$$a_{k-3}+3\,a_{k-2}+3\,a_{k-1}+a_k\cdots.$$

The process may be continued upwards, the coefficients being the sums of the coefficients in the line below. The top line becomes

$$a_1,\ a_1+a_2,\ a_2+2\,a_2+a_3,\ a_1+3\,a_2+3\,a_3+a_4,\ \cdots,$$
$$a_1+ma_2+\frac{m(m-1)}{1\cdot 2}\ a_3+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}a_4+\cdots.$$

For instance, we can find any term of the series of squares above in the form

$$1+3\,m+\frac{2\,m(m-1)}{2}=m^2+2\,m+1=(m+1)^2.$$

For sum add n & increase factorial another number

If we consider the squares to be the differences of a series whose first term is 0, the formula would give

$$1\,m + \frac{3\,m(m-1)}{2} + \frac{2\,m(m-1)(m-2)}{1\cdot 2\cdot 3}$$
$$= \tfrac{1}{6}\,m(m+1)(2\,m+1).$$

This method is very convenient for finding the formula for a sum of any series such that the differences somewhere become all alike.

It is evident that if we consider the series derived above as given and form its differences, the general term would be

$$a_1 + ma_2 + \frac{m(m-1)}{1\cdot 2}a_3 + \cdots - [a_1 + (m-1)a_2$$
$$+ \frac{(m-1)(m-2)}{1\cdot 2}a_3 + \cdots] = a_2 + (m-1)a_3 + \cdots.$$

Since this has the same form as the first general term, the next general term would be $a_3 + (m-2)a_4 + \cdots$ and so on down, eventually recovering the lines of differences.

Find the sum formula for the following, then write out the sum series, and all the differences, as in the example:

$$5 + 12 + 21 + 32 + \cdots$$
$$7 \quad 9 \quad 11$$
$$2 \quad 2$$

$$5\,m + \frac{7\,m(m-1)}{2} + \frac{2\,m(m-1)(m-2)}{6}$$
$$= \frac{m(m+1)2\,m+13)}{6}.$$

$$0 \quad 5 \quad 17 \quad 38 \quad 70 \cdots$$
$$5 \quad 12 \quad 21 \quad 32$$
$$7 \quad 9 \quad 11$$
$$2 \quad 2$$

Exercises are furnished by the series above given as exercises.

Fractions

3. In order to add fractions it is necessary to give them the same denominator. This is simply an instance of the formula $a(x + y + z + \cdots) = ax + ay + az + \cdots$. For an example consider the fractions

$$\tfrac{5}{6} + \tfrac{7}{8} \text{ or } \tfrac{40}{48} + \tfrac{42}{48} = \tfrac{1}{48}(40 + 42) = \tfrac{82}{48}.$$

As an example in letters let us add

$$\frac{3}{x-2} + \frac{5}{x-4} + \frac{-7}{x-5}.$$

The lowest expression all the denominators will divide without remainder is

$$(x-2)(x-4)(x-5).$$

Hence the first fraction must be written

$$\frac{3(x-4)(x-5)}{(x-2)(x-4)(x-5)}.$$

The second becomes

$$\frac{5(x-2)(x-5)}{(x-2)(x-4)(x-5)},$$

the third

$$\frac{-7(x-2)(x-4)}{(x-2)(x-4)(x-5)}.$$

If we add the numerators, we have

$$3(x-4)(x-5) + 5(x-2)(x-5) - 7(x-2)(x-4) = x^2 - 20x + 54.$$

If, now, we had given the fraction

$$\frac{x^2 - 20x + 54}{(x-2)(x-4)(x-5)},$$

could we recover the original fractions of which it is the sum? In order to do this we would have to find numbers A, B, C such that we could write as an identity

$$x^2 - 20x + 54 \equiv A(x-4)(x-5) + B(x-2)(x-5) + C(x-2)(x-4).$$

If such an identity exists, it will be satisfied by any value of x we substitute. Now it is evident that if we substitute $x = 2$, the terms containing B and C will be zero, since both have $x - 2$ as a factor. Hence we would have left the other parts with 2 substituted for x. Then

$$A = \left[\frac{x^2 - 20x + 54}{(x-4)(x-5)}\right]_{x=2} = 3.$$

In a like manner, if we make $x = 4$, we will have left only the second term on the right, and from this we have

$$B = \left[\frac{x^2 - 20x + 54}{(x-2)(x-5)}\right]_{x=4} = 5.$$

Finally,

$$C = \left[\frac{x^2 - 20x + 54}{(x-2)(x-4)}\right]_{x=5} = -7.$$

If we look these forms over, we see that each is obtained from the original fraction by omitting one factor at a time from the denominator, and substituting in the remaining expression the value of x which would make the omitted factor equal to 0. This procedure will evidently give the necessary coefficients whenever a given factor appears in all the terms except the one wanted. For instance, if we have to break up

$$\frac{5x^2 - 29x + 48}{(x-2)(x-3)^2},$$

we must be able to write the numerator in the form

$$5x^2 - 29x + 48 = A(x-2) + B(x-3)^2,$$

and A may be of the form $cx + d$, since the terms on the right may be as high as quadratics. If we set $x = 3$, we would have left the first term, but it would give us $3c + d = 6$, and we cannot find two coefficients from this single equation. However, if we set $x = 2$, we have

$$B = \left[\frac{5x^2 - 29x + 48}{(x-3)^2}\right]_{x=2} = 10.$$

This value of B may be substituted for it, giving now

$$5x^2 - 29x + 48 = [\quad\quad](x - 2) + 10(x - 3)^2.$$

Expanding the last term and transposing, we must have an expression which can be factored into $(x - 2)[\quad\quad]$. This is indeed the case, as it is found to be

$$(x - 2)[-5x + 21].$$

Hence we have $5x^2 - 29x + 48 = [-5x + 21](x - 2) + 10(x - 3)^2$, and the original fraction

$$\frac{5x^2 - 29x + 48}{(x - 2)(x - 3)^2} = \frac{-5x + 21}{(x - 3)^2} + \frac{10}{x - 2}.$$

We may now apply the Horner process and reduce $-5x + 21$ to the form $-5(x - 3) + 6$, so that the first fraction reduces to

$$\frac{-5}{(x - 3)} + \frac{6}{(x - 3)^2}.$$

It would have expedited matters to have written the trial form as

$$5x^2 - 29x + 48 = A(x - 2) + B(x - 2)(x - 3) + C(x - 3)^2.$$

Then we see we can find A and C both.

$$A = \left[\frac{5x^2 - 29x + 48}{(x - 2)}\right]_{x=3} = 6,$$

$$C = \left[\frac{5x^2 - 29x + 48}{(x - 3)^2}\right]_{x=2} = 10.$$

Substituting these and transposing both definite terms,

$$5x^2 - 29x + 48 - 6(x - 2) - 10(x - 3)^2 = B(x - 2)(x - 3),$$

or $$-5(x - 2)(x - 3) = B(x - 2)(x - 3),$$

and $$B = -5.$$

We have a similar situation when there is a factor in the denominator which is such that its factors would bring in irrationalities. While the solution could be carried through in terms of any irrational numbers, usually it is

not desired in that form. The procedure is similar to the one above. For instance,

$$\frac{7x^3 - 31x^2 - 9x + 111}{(x-2)(x-3)(x^2 - 2x - 5)} = \frac{5}{x-2} + \frac{3}{x-3} + \frac{-x+1}{x^2 - 2x - 5}.$$

Dropping the factor $x - 2$ and substituting in the remaining fraction $x = 2$, we get 5. Substituting 3 in the proper form, we have 3. The fraction with the quadratic denominator is now found by multiplying out, transposing, and dividing by $(x - 2)$ and $(x - 3)$,

$$7x^3 - 31x^2 - 9x + 111 = 5(x-3)(x^2 - 2x - 5) + 3(x-2)(x^2 - 2x - 5) + [\quad](x-2)(x-3)$$

or $$-x^3 + 6x^2 - 11x + 6 = [\quad](x-2)(x-3).$$

The bracket turns out to be $-x + 1$. These identities could, of course, be solved in other ways, and as a last resort we would substitute enough values for x to arrive at as many equations as we had unknown numbers, then solve by determinants.

EXERCISES

Break up the following fractions:

1. $\dfrac{7x - 17}{(x-2)(x-3)}$.
2. $\dfrac{x^2 - 14x + 9}{(x+1)(x-1)(x-2)}$.
3. $\dfrac{2x^2 - 6x - 14}{(x+4)(x-2)(x-3)}$.
4. $\dfrac{9x^2 + 16x - 24}{(x+2)(x+4)(x-3)}$.
5. $\dfrac{6x^2 - 30x - 6}{(x+4)(x-3)(x-6)}$.
6. $\dfrac{4x^2 - 8x + 3}{(x-1)^3}$.
7. $\dfrac{5x^2 + 2x - 7}{(x-2)^3}$.
8. $\dfrac{6x^3 + 2x + 3}{(x^2 - 2x - 4)^2}$.
9. $\dfrac{12x^2 - 16x + 4}{(x-2)(x^2+1)}$.
10. $\dfrac{17x^2 - 3x + 13}{x^3 - 1}$.
11. $\dfrac{6x^3 - 43x^2 + 134x - 141}{(x-2)(x-3)^3}$.
12. $\dfrac{9x^2 + 6x + 21}{(x-1)^2(x+2)^2}$.

13. $\dfrac{15x^2 - 151x + 374}{(x-4)(x-5)(x-6)}$.

15. $\dfrac{-6x^2 + 5x - 9}{(x^2+4)(x^2+x+1)}$.

14. $\dfrac{10x^3 - 55x^2 + 160x - 110}{(x-1)(x-2)(x-3)(x-4)}$.

16. $\dfrac{(a+b+c)x^2 - [(a+b+c)^2 - (ab+bc+ca)]x + b^2c + a^2b + c^2a}{(x-a)(x-b)(x-c)}$.

4. We may use these same forms to make up expressions that will give certain assigned values when we make x have certain given values. For instance, to find an expression which will give the values 2, 4, 6, 8 for $x = 1, 2, 3, 4$.

If we write down

$$\begin{aligned}&2(x-2)(x-3)(x-4)\\ &\quad + 4(x-1)(x-3)(x-4) + 6(x-2)(x-1)(x-4)\\ &\quad + 8(x-1)(x-2)(x-3),\end{aligned}$$

it is evident at once that when we make $x = 1$, every term but the first vanishes, and if we had a divisor under it of the form $(x-2)(x-3)(x-4)$, it would cancel the numbers from the same form in the term, and we would arrive at 2 as the value of the expression. A similar statement may be made as to the other terms. We can accomplish the part of providing the proper divisors by writing the final form

$$\frac{2(x-2)(x-3)(x-4) + 4(x-1)(x-3)(x-4) + 6(x-1)}{(x-2)(x-3)(x-4) + (x-1)(x-3)(x-4) + (x-1)}$$
$$\frac{(x-2)(x-4) + 8(x-1)(x-2)(x-3)}{(x-2)(x-4) + (x-1)(x-2)(x-3)}.$$

It is clear that when we make $x = 1$, there are three terms zero in both numerator and denominator, and we have 2; if we make $x = 2$, we have 4; if $x = 3$, we have 6; if $x = 4$, we have 8. The numerator should be reduced to a single term, as should the denominator. The final form is

$$\frac{20x^3 - 140x^2 + 300x - 192}{4x^3 - 30x^2 + 70x - 50}.$$

If this is tested by the four substitutions, we find that we have the proper forms. We could also have arrived at an integral form if we had divided each term by the proper value as a number. This would give

$$-\tfrac{1}{3}(x-2)(x-3)(x-4)+2(x-1)(x-3)(x-4)$$
$$-3(x-1)(x-2)(x-4)+\tfrac{4}{3}(x-1)(x-2)(x-3)$$
$$=\tfrac{1}{3}[6x]=2x.$$

This we could have seen at the start. In the usual case, however, the final form could scarcely be guessed. The fractional form is one that is used in building up the transformations that will give each of the roots of a Galois resolvent equation in terms of a single one of them. Since the product of all the factors may be found easily it will be simpler to use it for the formation of the fraction. The denominator is then the first Sturm function. The numerator can be found easily by dividing the product in turn by each factor and, after multiplying by the value corresponding, adding the results. In the example above the product of all the factors is

$$F = x^4 - 10x^3 + 35x^2 - 50x + 24.$$

The first Sturm function is then $4x^3 - 30x^2 + 70x - 50$. As a further example let us find the fraction that will give the values 3, -1, 2, -4 for $x = 1, 2, 3, 4$. We have F as just written. The numerator is found as follows:

$$\begin{array}{l} 1-10+35-50+24(1 \\ \quad\quad 1-9+26 \\ \hline 1-9+26-24 \\ \quad\quad\quad\quad 3 \\ \hline 3-27+78-52 \end{array} \qquad \begin{array}{l} 1-10+35-50+24(2 \\ \quad\quad 2-16+38 \\ \hline 1-8+19-12 \\ \quad\quad\quad\quad -1 \\ \hline -1+8-19+12 \end{array}$$

$$\begin{array}{l} 1-10+35-50+24(3 \\ \quad\quad 3-21+42 \\ \hline 1-7+14-8 \\ \quad\quad\quad\quad 2 \\ \hline 2-14+28-16 \end{array} \qquad \begin{array}{l} 1-10+35-50+24(4 \\ \quad\quad 4-24+44 \\ \hline 1-6+11-6 \\ \quad\quad\quad\quad -4 \\ \hline -4+24-44+24 \end{array}$$

Adding, the fraction is

$$\Phi = \frac{-9x^2 + 43x - 32}{4x^3 - 30x^2 + 70x - 50}.$$

EXERCISES

1. Find an integral function which will have the values $-2, -4, 3, 5$ for $x = -1, 1, 2, 4$.

2. Find an integral function which will have the values $4, 2, 3, 6, 8$ for $x = 1, 2, 3, 4, 5$.

3. Find a fractional function which will give for the values of x, $-2, -1, 1, 2$, the values $4, 2, 2, 4$.

4. Find a fractional function which will give for the values of x, the values ϕ

$$\text{for } x = 2, 3, -5, -2, -3, 5,$$
$$\phi = 3, -5, 2, 5, -2, -3,$$

Ans. $$\Phi = \frac{-38x^4 - 56x^3 + 722x^2 + 1064x - 2700}{6x^5 - 152x^3 + 722x}.$$

Find the function when the values are $-2, -3, 5, 2, 3, -5$.

$$\Phi = \frac{-76x^4 + 1444x^2 - 5400}{6x^5 - 152x^3 + 722x}.$$

As further exercises find the functions when the values are in order

$$-5, 2, 3, -3, 5, -2,$$
$$-3, 5, -2, -5, 2, 3,$$
$$5, -2, -3, 3, -5, 2.$$

The six functions obtained (including the function $\Phi = 1x$) are such that if the values obtained from the original set for any one be substituted in the function for any other arrangement, then one of the six arrangements will be obtained. The six functions are then said to form a *group* for the given six values.

The six given values are the differences of the three numbers 1, 3, 6, taken in every way. These are the roots of the cubic

$$x^3 - 10x^2 + 27x - 18 = 0.$$

The function $F = x^6 - 38x^4 + 361x^2 - 900 = 0$ gives the Galois resolvent sextic, for the coefficients $l = 1$, $m = -1$,

$n = 0$. The six functions above give the rational expression of each of the roots in terms of each of the others. If each is substituted in the others and reduced by $F = 0$, we arrive at the same six. This is the group property.

Tactic Arrangements

5. The coefficients we have used above for binomials find other uses, which we will now consider, and we will at the same time find the *polynomial coefficients*. We may arrive most easily by considering the complete product of the binomial expansion. For instance, if we multiply together $(x + a)(x + a)$ and keep all the terms without reducing, we shall have

$$xx + xa + ax + aa.$$

If now we multiply again by $x + a$, we shall have the expansion

$$xxx + xxa + xax + axx + xaa + axa + aax + aaa.$$

We can look at these terms as arrangements of the letters; in the first and last terms, all the letters are the same; in the second, third, and fourth, there are two alike and one different; and in the fifth, sixth, and seventh, there are two alike and the other different. Grouping together those that are alike, we would have

$$xxx + \frac{3!}{2!1!}xxa + \frac{3!}{2!1!}aax + aaa.$$

6. The problem is the same as if we were trying to put two kinds of things into three boxes. We may put the same thing, x, in all of the boxes in one way, or two boxes filled with x and one with a, in three ways, or two with a, and one with x in three ways, or finally all a in one way. If we write out the expansion of

$$(1 + 1)^3,$$

we have only the coefficients, and the terms of the expansion will give us the number of ways we can select the

various combinations. The coefficients are the numbers which indicate the different ways in which certain things can be arranged in order, of which one group are all alike, say x, and the other group are all alike, say a. The general formula for such a coefficient is for $n = x_1 + x_2 + \cdots + x_c$,

$$\frac{n!}{x_1!\,x_2!\cdots x_c!}.$$

We get this formula by considering that if all the n things were different, they could be arranged in $n!$ ways. But when we make x_1 of them the same, we have evidently counted the $x_1!$ ways they would appear as different when they should have been counted but once. Consequently we must divide by this number, and similar factorials for each of the others. As an instance the letters in *tennessee* can be arranged in

$$\frac{9!}{1!\,4!\,2!\,2!} = 3780 \text{ ways.}$$

This would also be the coefficient in

$$(t + e + n + s)^9$$

of the term $t^1e^4n^2s^2$.

0! = 1

EXERCISES

1. In how many ways can the letters in *illinois* be arranged? in *cincinnati?* in *mississippi?* in *banana?* in *surface?*

2. If we have nine objects from which we are to put one in each of four boxes, in how many ways can we select an object for the first box? After this, in how many ways may an object be selected for the second box? In how many ways can the two boxes receive their objects? the four?

3. Write out the different designs we could make, each design of three vertical ribbons, if we have seven colors of ribbons to select from.

4. Write out all the expanded terms, with no reductions, for

$$(a + b + c)^2.$$

5. Write out the expanded terms for $(a + b + c + d)^5$ which have one letter twice, another twice, and a third once.

6. The terms of the different ways in which four things could be put into five boxes are given by the expansion of

$$(1 + 1 + 1 + 1)^5.$$

What is the total number, and how many of these would have the same thing in two boxes, another thing in each of two more, and a third thing in the last?

7. We may apply these ideas to the question of averages and predictions based on averages. Such predictions are called in mathematics *probabilities*. For instance, if the record shows that A wins three games of chess out of four and loses one, when he plays B, we say his record is shown by the fractions

$$\tfrac{3}{4} + \tfrac{1}{4}.$$

If we arrange to have A and B play several games, the two fractions would indicate what we might expect the result to be. If, now, we arrange to have another set played, we would expect the same fractions to indicate the second result. If we write out the expansion of

$$(\tfrac{3}{4} + \tfrac{1}{4})^2,$$

the term

$$(\tfrac{3}{4})^2$$

would give us the fraction indicating the number of games A would win in the first set combined with the number he would win in the second set. The term

$$\frac{2!}{1!\,1!}\ \frac{3}{4} \times \frac{1}{4}$$

would indicate the combination of winning in one set and losing in the other. The term

$$(\tfrac{1}{4})^2$$

would indicate the combinations of losing in both sets.

Similar expansions would be used to find the predicted averages, or probabilities, for other problems. For in-

stance, if a ball team has won 6 games out of every 10, lost 3, and tied 1, their fractional averages would be

$$\tfrac{6}{10} + \tfrac{3}{10} + \tfrac{1}{10}.$$

If, now, they play 5 sets of 10 games each, we may expect the combinations given by

$$[\tfrac{6}{10} + \tfrac{3}{10} + \tfrac{1}{10}]^5.$$

For instance, the average we would expect of their winning all games in 2 of the 5 sets, losing in 2, and tying in 1 would be

$$\frac{5!}{2!2!1!}\left(\frac{6}{10}\right)^2\left(\frac{3}{10}\right)^2\left(\frac{1}{10}\right).$$

EXERCISES

1. If A has a record of 3 games out of 5 with B and 2 games out of 3 with C, what would be his combined set of averages with both, if he plays a set with each? That is, what would be his fractional average of winning with both, winning with B and losing with C, winning with C and losing with B, losing with both?

2. If a group of computers have an average record that each will solve correctly 8 problems out of 10, and they all work a set of 4 problems, how many can be expected to get all 4, how many will have 3 correct and 1 wrong, how many will have 2 right and 2 wrong?

3. If a coin is tossed a thousand times and it is found that half of the times it came heads up, and half tails up, what would be the expected average if 5 coins are tossed a thousand times, that they would all be heads, 4 would be heads, 3 heads, 2 heads, 1 head, all tails?

4. If a box contains 10 balls and 2 are drawn out, how many different pairs could be drawn? If 5 of the balls are white and 5 black, in how many ways could a pair of white balls happen? What would be the expected average of white pairs in several thousand drawings? of pairs of one white and one black?

5. If many drawings are made, what would be the fractional average of those in which there were more white than black balls?

8. There are many types of arrangements of a different kind than in a row. As, an instance, the student may write out the different ways in which fifteen school girls can take a promenade, marching in triads, such that the five triads will change each successive day. That is, if two girls are together on one day, they will not be together again in a triad. This problem has led to some very interesting investigations.

Another interesting problem is to arrange a set of numbers in the cells in a square so that the sums of the rows and of the columns will give the same number. Such arrangements are called *magic squares.* The explanation of their construction will be found in various encyclopedias and special books devoted to the matter. One instance will be given. If we put the nine numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 in a square arrangement, the sum of the three columns would have to be 45 and if each column had the same sum, each would have to add up to 15. Now the sets of three numbers whose sum is 15 are $1+5+9, 1+6+8, 2+4+9, 2+5+8, 2+6+7, 3+4+8, 3+5+7, 4+5+6$. If we start with 1, we may arrange the square thus

$$\begin{array}{ccc} 1 & 5 & 9 \\ 6 & & \\ 8 & & \end{array}$$

Since 3 does not occur in a combination with 6, it must go on the bottom line with 8, and with it must go 4. This gives us

$$\begin{array}{ccc} 1 & 5 & 9 \\ 6 & 7 & 2 \\ 8 & 3 & 4 \end{array}$$

We could not put 4 in the second column, as its only combination with 5 is with 6, which is not in that column.

The student may try what can be done with the numbers from 1 to 16. Each column must add up to 34, as well as each row.

The entire subject of arrangements is called *tactic*.

9. One of the most important parts of algebra is the study of identities. We are concerned here with structure, which is the real topic in algebra. In the various forms of structure we find examples of order and composition in the design; symmetry of various types; repetition and cadence in the rhythm; harmony of forms; and always the unity which is at present everywhere in Mathematics and which ultimately makes all Mathematics the expression of a single Universal Unity.

> "I am the poet's dream,
> The harmony
> Of all the music and of all the spheres,
> The painter's visionary gleam,
> The illumined page,
> The storied lore and heritage
> Of all the years."—*Alice Fay*

The interested student is recommended to study: Algebraic Analysis, by Wentworth, McClellan, and Glashan.

CHAPTER VIII

GROUPS, INVARIANTS

1. We will consider briefly some ideas we have mentioned earlier. To start with an example. Consider the fraction

$$y = \frac{4x + 7}{3x - 4}.$$

If we make a new fraction,

$$z = \frac{4y + 7}{3y - 4},$$

which is of the same form in y as the first is in x, we may substitute the value of y, giving

$$\frac{4\left(\frac{4x + 7}{3x - 4}\right) + 7}{3\left(\frac{4x + 7}{3x - 4}\right) - 4},$$

which reduces to x. We have a remarkable property here, then, in that if we construct the fraction for any value of x, and then construct the same fraction for the result obtained in the first case, we will have as the final result the original value x. To see what sort of fraction of this kind would have this property, let

$$y = \frac{ax + b}{cx + d}, \quad z = \frac{ay + b}{cy + d} = \frac{x(a^2 + bc) + b(a + d)}{x(c)(a + d) + (bc + d^2)}.$$

If, now, we set $z = x$, for any value of x, we must have

$$b(a + d) = 0, \text{ giving } b = 0 \text{ or } a + d = 0.$$

The case $b = 0$ would make it necessary for $c = 0$, and

the original fraction would merely be x itself. Hence we must take

$$a + d = 0,$$

and this satisfies all the conditions, for we can easily prove that if

$$y = \frac{ax + b}{cx - a},$$

then $z = x$, whatever a, b, c are.

We call the fraction above a *linear fractional substitution*, if we are more interested in the coefficients and the form of the fraction than we are in the value. If the fractional substitution

$$\frac{ax + b}{cx + d}$$

is such that $a = -d$, a repetition of the substitution brings back the original value, and the fractional substitution is said to be of *order* 2.

2. If we now consider the substitution

$$y = \frac{3x - 13}{x - 4},$$

we find that

$$z = \frac{3y - 13}{y - 4} = \frac{4x - 13}{x - 3},$$

so that this substitution is not of order 2. But if we construct

$$w = \frac{3z - 13}{z - 4} = x,$$

we see that three repetitions produce the original value x. Hence this substitution is of order 3. The condition is that

$$a^2 + ad + d^2 + bc = 0.$$

The repetition of the substitution is easily found by interchanging a and d with a change in their signs also. For

instance, if we represent the substitution by S and its repetition by S^2, if

$$S(x) = \frac{6x - 7}{13x + 5},$$

$$S^2(x) = \frac{-5x - 7}{13x - 6},$$

$$S^3(x) = x.$$

3. When the successive results of substitutions give only a finite number of expressions, they are said to be a finite group of substitutions. An example is the set

$$S_1(x) = x, \qquad S_2(x) = \frac{21x - 88}{5x - 21}, \qquad S_3(x) = \frac{35x - 153}{5x - 35}$$

$$S_4(x) = \frac{30x - 133}{7x - 31}, \qquad S_5(x) = \frac{31x - 133}{7x - 30}$$

$$S_6(x) = \frac{14x + 65}{13x - 14}.$$

If we look these over, S_1 gives an identical substitution, S_2, S_3, S_6 are of order 2, and S_4, S_5 are of order 3, and $S_4^2 = S_5$. Further,

$$S_3(S_4[x]) = S_2(x), \qquad S_3(S_5[x]) = S_6(x).$$

4. The two substitutions,

$$\frac{ax + b}{cx + d} \quad \text{and} \quad \frac{-dx + b}{cx - a}$$

are called *inverse* to each other, since either followed by the other gives identically what we started with. The inverse of S is usually written

$$S^{-1}(x).$$

Since if S is of order 3, S^2 is its inverse, we can see now why the forms

$$\frac{30x - 133}{7x - 31} \quad \frac{31x - 133}{7x - 30} \text{ are inverse.}$$

The table showing the effect of successive substitutions is called a *multiplication table for the group.* For the example above the table is as follows

	S_1	S_2	S_3	S_4	S_5	S_6
S_1	S_1	S_2	S_3	S_4	S_5	S_6
S_2	S_2	S_1	S_5	S_6	S_3	S_4
S_3	S_3	S_4	S_1	S_2	S_6	S_5
S_4	S_4	S_3	S_6	S_5	S_1	S_2
S_5	S_5	S_6	S_2	S_1	S_4	S_3
S_6	S_6	S_5	S_4	S_3	S_2	S_1

To read the table, the substitution on the left, operating after the one at the top of a column, gives the substitution on the line of the first and under the second. For instance,

$$S_4 S_2 = S_3.$$

5. We also arrive at groups when we study permutations. The six permutations of the letters a, b, c may be considered as the six substitutions below,

$$\begin{array}{cccccc} abc & abc & abc & abc & abc & abc \\ abc & acb & bac & bca & cab & cba \end{array}$$

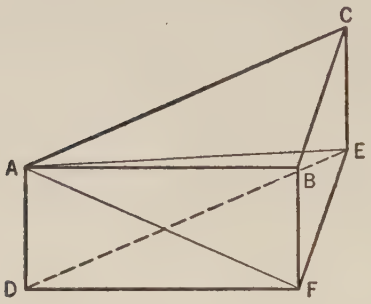
and these correspond to the linear fractional substitutions by using the letters

$$S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \quad S_6.$$

If we perform these permutations in succession, we would arrive at the same table as we have above. For instance,

$$\begin{array}{l} abc \\ S_4 S_2 \text{ means } acb, \text{ and this is } S_3. \\ bac \end{array}$$

6. We can find another example of the same group in certain problems of geometry. Suppose we consider the routes along the edges of the polyhedron in the figure from a given corner, the one marked A, then we will arrive at the same group as above.



We will call the route AD, S_3, and the route AB, S_4, then BF is S_3, CE is S_3, BC is S_4, CA is S_4, DE is S_4, EF is S_4, FD is S_4. Starting at A, we would have, then, AA is S_1, AD is S_3, AB is S_4, AF is S_6, AC is S_5, AE is S_2. Starting at any other corner, we have similar expressions, as BB is S_1, BC is S_4, BF is S_3, BA is S_5, BE is S_6, BD is S_2. If we now start anywhere and compound these routes, we will arrive at the multiplication table above. Groups which have the same multiplication table, whatever the letters may mean, are called *isomorphic*.

7. Substitutions are not always linear; as, an example,

$$x_1 = \frac{x}{x^2 + y^2}, \qquad y_1 = \frac{y}{x^2 + y^2}.$$

In these we have squares in the denominators. If we invert them by solving for x and y, we find the same expressions, with x_1 and y_1 in the fractions.

8. In the last example given if we construct $\frac{y}{x}$, we find

$$\frac{y_1}{x_1} = \frac{y}{x}.$$

We call the expression $\frac{y}{x}$ an *invariant* for this transformation. Invariants have occurred before in previous chapters. For instance, any symmetric expression in the

three letters a, b, c is an invariant for the six permutations in 5. As, for example, $a^3 + b^3 + c^3$ is unchanged by interchanging the letters. Invariants play a very important part in the study of expressions containing the roots of an equation. Groups also are important, for the work of Galois was based upon the study of the groups of the transformations which converted a given irrational root of a Galois resolvent into the other roots. The cubic, for instance, is connected with the roots of the permutation group in 5.

The permutations of n letters can be divided into two classes: those with an even number of inversions, and those with an odd number. This is why a determinant has as many positive terms as negative terms. If all the terms are subjected to permutations of the subscripts, this would be equivalent to interchanging rows in the square form, and if the letters are interchanged, it is equivalent to interchanging the columns. As these changes produce only a change of sign if the permutation used is odd, and no change of sign if it is even, we see that a determinant is an invariant for even permutations of the rows or columns. As all even permutations form a group x, which is called the alternating group on n letters, we can say in brief that a determinant is an invariant for the alternating group of permutations of its rows or its columns. The subject is very important, but goes into higher mathematics.

9. We find in groups and invariants another instance of the esthetic character of mathematics, for a group gives a definite design to the things operated on, and the invariants give the character of permanency needed in any composition, the character of unity through all the changes. The symmetry also in groups is of a more complicated kind than that found in the usual right and left symmetry, which corresponds merely to a simple

operation of order 2. So that through groups we enter the realm of what we may call higher symmetries.

10. If we consider a group carefully, we also see in it examples of rhythm in the repetition of the operations, and also in the effects of the operators on the parts of the group which themselves make smaller groups, called *sub-groups*. We cannot enter into this subject, as it would take us too far. It is suggested to the ambitious student.

Invariants

11. An invariant is an expression which is not changed by some substitution on the letters in the expression. For instance, if we let l and m be two fractions such that $l^2 + m^2 = 1$, and set

$$x = lu + mv,$$
$$y = mu - lv,$$

then the expression $x^2 + y^2$ is invariant under the change from x, y to u, v, for

$$x^2 + y^2 = (lu + mv)^2 + (mu - lv)^2$$
$$= u^2(l^2 + m^2) + v^2(l^2 + m^2) = u^2 + v^2.$$

If the expression in the original letters comes out the same in the new letters multiplied by some number which depends only on the coefficients in the substitution, then we call the expression a *relative invariant*. For instance, if we consider the discriminant of the quadratic

$$ax^2 + bx + c = 0,$$
$$\Delta = b^2 - 4ac,$$

and for x we write,

$$\frac{lu + m}{pu + q},$$

we have, after multiplying out both sides,

$$Au^2 + Bu + C = 0,$$

where

$$A = al^2 + blp + cp^2,$$
$$B = 2\,alm + blq + bmp + 2\,cpq,$$
$$C = am^2 + bmq + cq^2.$$

If, now, we construct the discriminant of the quadratic in u, we will have, after reducing it,

$$\Delta' = \Delta(lq - mp)^2.$$

We see, then, that if we choose l, m, p, q, so that $lq - mp = 1$, $\Delta' = \Delta$. As an example we could let

$$x = \frac{3\,u + 7}{2\,u + 5}.$$

Hence Δ is a relative invariant, and in case the expression

$$\begin{vmatrix} l & m \\ p & q \end{vmatrix}$$

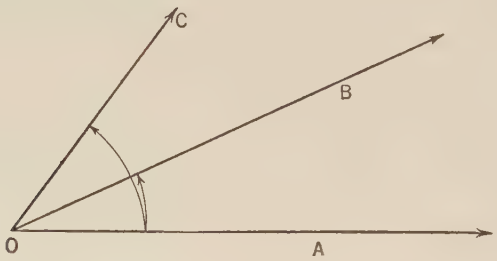
(this determinant is called the *modulus* of the substitution) is unity, Δ is an invariant, that is, for this substitution.

Δ is the only invariant of the quadratic. Higher polynomials have increasing numbers of invariants. Expressions which would contain x, and would give invariant forms, are called *covariants*. The study of these forms is very interesting. Much of its development is due to Sylvester.

CHAPTER IX

HYPERNUMBERS

1. In treating the complex number we defined number in general as the ratio of two directed segments of lines in a plane. If we extend the same idea to lines in space, we arrive at a system of numbers which bring in entirely new properties.



Let the ratio of OB to OA be represented by α, of OC to OA by β, OC to OB by γ, then we can write

$$OC = \gamma \cdot OB = \gamma \cdot \alpha OA = \beta \cdot OA.$$

If, then, we consider the ratios alone, we could write

$$\beta = \gamma \cdot \alpha.$$

If we notice that α, β, γ are complex numbers exactly of the same nature as those previously studied, but related to different planes, we shall see that we are now concerned with an extension of complex numbers which include them, just as they included reals. These new numbers are *hypernumbers*, and are called, for this particular type, *quaternions*. They were the creation of Sir William Rowan Hamilton.

2. If the three lines OA, OB, OC are mutually perpendicular, the numbers they define are usually represented by i, j, k.

$$i = OC : OB,$$
$$j = OA : OC,$$
$$k = OB : OA.$$

It is not hard to see that we have, then,

$$ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad ik = -j,$$
$$i^2 = -1, j^2 = -1, k^2 = -1.$$

These are called the defining equations for quaternions, because all quaternions can be written in terms of $1, i, j, k$, just as complex numbers may be written in terms of 1 and $\sqrt{-1}$. If the student will look over the equations of multiplication just written, he will find the two, $ij = k$, $ji = -k$. This is a new property of numbers, for so far, the product of two numbers was the same in whatever order they were taken. In any case where we have

$$ab = ba,$$

whatever they may mean, we say they have *commutative multiplication*. If this is not the case, then they are *non-commutative*. In the one case in which

$$ab = -ba,$$

the multiplication is called *skew* or *polar*.

3. We may now notice the *laws of combination* of numbers. We have for addition usually the commutative property, that is,

$$a + b = b + a.$$

Subtraction is not commutative, for

$$a - b \neq b - a.$$

Multiplication is commutative for real and complex numbers, but not for quaternions. Division is not commutative.

We also have a property called *distributive*. This relates to two different processes, one being distributive as to the other. It is true of multiplication as to addition or subtraction. That is,

$$a(b + c) = ab + ac.$$

There is another property of a single process, called *associative*. This relates to the grouping of the operation, but not to a change of order. It is a property of addition, that is,

$$a + (b + c) = (a + b) + c.$$

It is not true of subtraction, for

$$a - (b - c) \text{ is not } (a - b) - c.$$

It is a property of multiplication. That is,

$$a(bc) = (ab)c.$$

The distributive and associative properties hold for quaternion addition and multiplication, just as for their special cases of real or complex numbers. There are types of number, however, for which one or both properties fail.

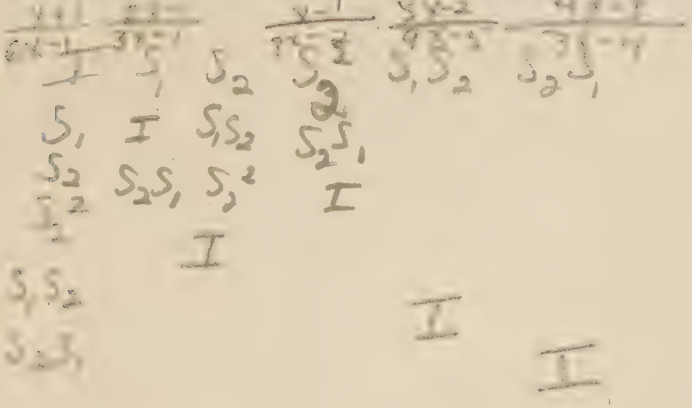
4. We may notice that the different units of quaternions form in their multiplication a table which has group properties. The table would be

	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	i	-1	k	$-j$	$-i$	1	$-k$	j
j	j	$-k$	-1	i	$-j$	$+k$	1	$-i$
k	k	j	$-i$	-1	$-k$	$-j$	i	1
-1	-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-k$	$-j$	i	1	k	j	$-i$	-1

Considered as a group of substitutions or operations, this is called the quaternion group. It has many interesting properties.

5. Numbers which are like quaternions are called *hypernumbers*. They furnish an endless series of new kinds of numbers, which are called not only hypernumbers but *qualitative, or relative,* numbers. Each sign, like i, j, k, gives a new character to the number, the arithmetic part retaining its former properties. Magnitude has to do with the arithmetic character. Kind or quality has to do with what we may call the strictly algebraic character. The structure of systems of hypernumbers is a fundamental part of algebra.

6. We have now made the acquaintance of the various parts of algebra in a very elementary way, and the continuation of the study would belong to higher algebra. Arithmetic is continued in the "Theory of Numbers." Equations in one unknown is continued in "Theory of Equations." Equations in more than one unknown is continued in what usually is called "Modern Algebra." Gropus and Invariants and Hypernumbers retain the same name in their continuations. Everywhere we find the esthetic features prominent, if we look for them.



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*	I	N	IN
*	I	N	IN
I	*	IN	N
N	IN	*	I
IN	N	I	*

Logarithms

$1 + 1 + 1 + \cdots$ (n terms) $= n$ Sum $= n$

$1 + 2 + 3 + \cdots$ (n terms) $= \frac{1}{2}n(n+1)$ Sum $= \frac{1}{2}n(n+1)$

$1\cdot2 + 2\cdot3 + 3\cdot4 + \cdots = \frac{1}{3}n(n+1)(n+2)$

$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$

$a + (a+d) + (a+2d) + \cdots + a + (n-1)d = na + \frac{1}{2}n(n-1)d$

u_1	u_n	S_n
u_1	$u_1 + (n-1)d$	$nu_1 + \frac{1}{2}n(n-1)d = S_n = \frac{n}{2}(u_1 + u_n)$

$1 + 1 + 1 + 1 + \cdots + 1 = n$

$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$

$1\cdot2 + 2\cdot3 + 3\cdot4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

$an^2 + bn + c$

$an(n+1) + (b-a)n + c$

$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

u_1 u_2

$1^2 \quad 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

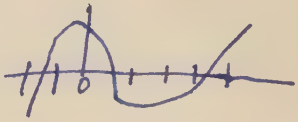
$2^2 + 4^2 + \cdots + (2n)^2 = \frac{2}{3}n(n+1)(2n+1)$

$x^3 - 2x^2 - 4x + 1 = 0$ Limits 9 times

$3x^2 - 4x - 4$

$32x + 1$

-2	-1	$-\infty$	0	∞	1	3	4
−	+	−	+	+	−	−	+
+	+	+	−	+	−	+	+
−	−	−	−	+	+	+	+
+	+	+	+	+	+	+	+
3	2	3	2	0	−1	1	0



$9-18-36+9 \quad \lfloor 3-4-4$

$9-12-12$

$-6-24+9$

$-6+8+8$

$-32+1$

$3-6-12+3 \quad \lfloor 3-4-4$

$3-4-4$

$-2-8+3$

$=2$

$$\frac{6x^3-43x^2+134x-141}{(x-2)(x-3)^3} \equiv \frac{-3}{x-2} + \frac{[9x^2-52x+111]}{(x-3)^3} \qquad \frac{9}{x-3}+\frac{2}{(x-3)^2}+\frac{36}{(x-3)}$$

$6x^3-43x^2+134x-141 \equiv -3(x-3)^3+[\quad](x-2)$

$6-43+134-141 \lfloor 2$

$12-62+144$

$6-31+72/+3$

-1

Regno always divisor in more than 2 col.

$$\begin{vmatrix} 4 & 3 & 2 \\ 5 & 4 & 3 \\ 3 & 2 & 8 \end{vmatrix} = 7$$

$$= \begin{vmatrix} 4 & 12 & 8 \\ 5 & 16 & 12 \\ 3 & 8 & 32 \end{vmatrix}$$

$$\begin{vmatrix} 4 & 0 & 0 \\ 5 & 1 & 2 \\ 3 & -1 & 26 \end{vmatrix}$$

$$\frac{4}{28} = 7$$

$$\begin{vmatrix} 23 & -19 \\ 17 & -13 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -1 \\ 17 & -3 \end{vmatrix} \quad \frac{-13}{4}$$

$$\begin{vmatrix} 1 & 2 \\ -1 & 26 \end{vmatrix}$$

$$4x - 3y - z = 0$$
$$5x + 2y - 6z = 1$$
$$7x - 5y - 5z = 1$$

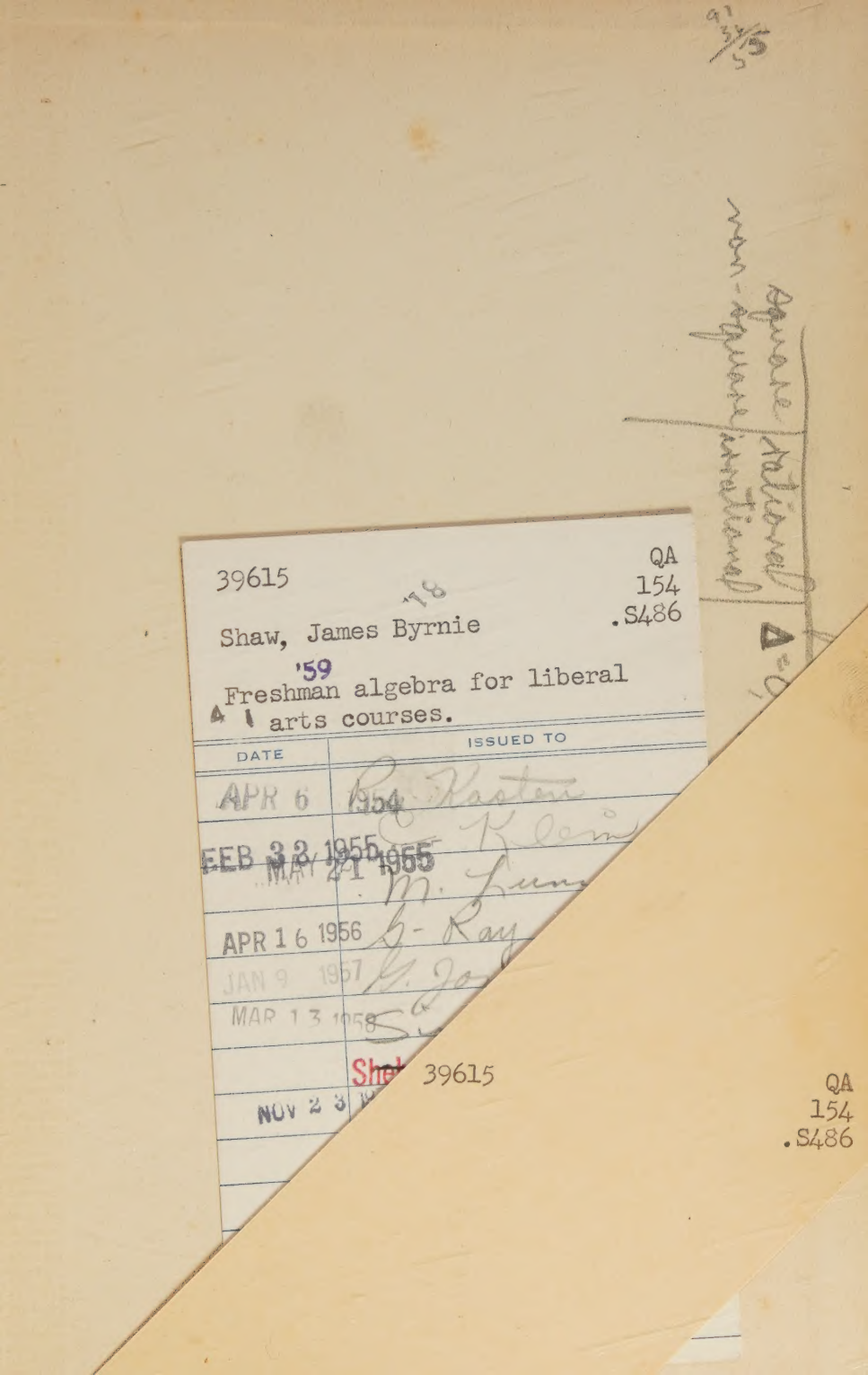
$$x = \frac{\begin{vmatrix} 0 & -3 & -1 \\ 1 & 2 & -6 \\ 1 & 5 & -5 \end{vmatrix}}{\begin{vmatrix} 4 & -3 & -1 \\ 5 & 2 & -6 \\ 7 & 5 & -5 \end{vmatrix}} \quad 6$$

$$\frac{\begin{vmatrix} 1 & 14 \\ 1 & -16 \end{vmatrix}}{6} = \frac{6}{6} = 1$$

$$y = \frac{\begin{vmatrix} 4 & 0 & -1 \\ 5 & 1 & -6 \\ 7 & 1 & 5 \end{vmatrix}}{6}$$

$$\frac{\begin{vmatrix} 14 & -19 \\ 14 & -13 \end{vmatrix}}{6} = \frac{6}{6} = 1$$

$$z \frac{\begin{vmatrix} 4 & -3 & 0 \\ 5 & 2 & 1 \\ 7 & -1 & 1 \end{vmatrix}}{6} = 1$$



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